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Differential Geometry

Connections, Curvature, and Characteristic Classes

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Connections, Curvature, and Characteristic Classes

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Preface

Differential geometry has a long and glorious history. As its name implies, it is the study of geometry using differential calculus, and as such, it dates back to Newton and Leibniz in the seventeenth century. But it was not until the nineteenth century, with the work of Gauss on surfaces and Riemann on the curvature tensor, that differential geometry flourished and its modern foundation was laid. Over the past one hundred years, differential geometry has proven indispensable to an understanding of the physical world, in Einstein's general theory of relativity, in the theory of gravitation, in gauge theory, and now in string theory. Differential geometry is also useful in topology, several complex variables, algebraic geometry, complex manifolds, and dynamical systems, among other fields. It has even found applications to group theory as in Gromov's work and to probability theory as in Diaconis's work. It is not too far-fetched to argue that differential geometry should be in every mathematician's arsenal.

The basic objects in differential geometry are manifolds endowed with a metric, which is essentially a way of measuring the length of vectors. A metric gives rise to notions of distance, angle, area, volume, curvature, straightness, and geodesics. It is the presence of a metric that distinguishes geometry from topology. However, another concept that might contest the primacy of a metric in differential geometry is that of a connection. A connection in a vector bundle may be thought of as a way of differentiating sections of the vector bundle. A metric determines a unique connection called a Riemannian connection with certain desirable properties. While a connection is not as intuitive as a metric, it already gives rise to curvature and geodesics. With this, the connection can also lay claim to be a fundamental notion of differential geometry.

Indeed, in 1989, the great geometer S. S. Chern wrote as the editor of a volume on global differential geometry [5], "The Editor is convinced that the notion of a connection in a vector bundle will soon find its way into a class on advanced calculus, as it is a fundamental notion and its applications are wide-spread."

In 1977, the Nobel Prize-winning physicist C. N. Yang wrote in [23], "Gauge fields are deeply related to some profoundly beautiful ideas of contemporary

mathematics, ideas that are the driving forces of part of the mathematics of the last 40 years, ..., the theory of fiber bundles." Convinced that gauge fields are related to connections on fiber bundles, he tried to learn the fiber-bundle theory from several mathematical classics on the subject, but "learned nothing. The language of modern mathematics is too cold and abstract for a physicist" [24, p. 73].

While the definition and formal properties of a connection on a principal bundle can be given in a few pages, it is difficult to understand its meaning without knowing how it came into being. The present book is an introduction to differential geometry that follows the historical development of the concepts of connection and curvature, with the goal of explaining the Chern–Weil theory of characteristic classes on a principal bundle. The goal, once fixed, dictates the choice of topics. Starting with directional derivatives in a Euclidean space, we introduce and successively generalize connections and curvature from a tangent bundle to a vector bundle and finally to a principal bundle. Along the way, the narrative provides a panorama of some of the high points in the history of differential geometry, for example, Gauss' Theorema Egregium and the Gauss–Bonnet theorem.

Initially, the prerequisites are minimal; a passing acquaintance with manifolds suffices. Starting with Section 11, it becomes necessary to understand and be able to manipulate differential forms. Beyond Section 22, a knowledge of de Rham cohomology is required. All of this is contained in my book *An Introduction to Manifolds* [21] and can be learned in one semester. It is my fervent hope that the present book will be accessible to physicists as well as mathematicians. For the benefit of the reader and to establish common notations, we recall in Appendix A the basics of manifold theory. In an attempt to make the exposition more self-contained, I have also included sections on algebraic constructions such as the tensor product and the exterior power.

In two decades of teaching from this manuscript, I have generally been able to cover the first twenty-five sections in one semester, assuming a one-semester course on manifolds as the prerequisite. By judiciously leaving some of the sections as independent reading material, for example, Sections 9, 15, and 26, I have been able to cover the first thirty sections in one semester.

Every book reflects the biases and interests of its author. This book is no exception. For a different perspective, the reader may find it profitable to consult other books. After having read this one, it should be easier to read the others. There are many good books on differential geometry, each with its particular emphasis. Some of the ones I have liked include Boothby [1], Conlon [6], do Carmo [7], Kobayashi and Nomizu [12], Lee [14], Millman and Parker [16], Spivak [19], and Taubes [20]. For applications to physics, see Frankel [9].

As a student, I attended many lectures of Phillip A. Griffiths and Raoul Bott on algebraic and differential geometry. It is a pleasure to acknowledge their influence. I want to thank Andreas Arvanitoyeorgos, Jeffrey D. Carlson, Benoit Charbonneau, Hanci Chi, Brendan Foley, George Leger, Shibo Liu, Ishan Mata, Steven Scott, and Huaiyu Zhang for their careful proofreading, useful comments, and errata lists. Jeffrey D. Carlson in particular should be singled out for the many excellent pieces of advice he has given me over the years. I also want to thank Bruce Boghosian for

helping me with Mathematica and for preparing the figure of the Frenet–Serret frame (Figure 2.5). Finally, I am grateful to the Max Planck Institute for Mathematics in Bonn, National Taiwan University, and the National Center for Theoretical Sciences in Taipei for hosting me during the preparation of this manuscript.

Medford, MA, USA
April 2017

Loring W. Tu

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A Circle Bundle over a Circle by Lun-Yi Tsai and Zachary Treisman, 2010.
Welded stainless steel and latex sheeting.

Chapter 1

Curvature and Vector Fields

By a manifold, we will always mean a smooth manifold. To understand this book, it is helpful to have had some prior exposure to the theory of manifolds. The reference [21] contains all the background needed. For the benefit of the reader, we review in Appendix A, mostly without proofs, some of the definitions and basic properties of manifolds.

Appendix A concerns smooth maps, differentials, vector fields, and differential forms on a manifold. These are part of the *differential topology* of manifolds. The focus of this book is instead on the *differential geometry* of manifolds. Now a manifold will be endowed with an additional structure called a ***Riemannian metric***, which gives a way of measuring length. In differential geometry, the notions of length, distance, angles, area, and volume make sense, whereas in differential topology, since a manifold can be stretched and still be diffeomorphic to the original, these concepts obviously do not make sense.



Bernhard Riemann

(1826–1866)

Some of the central problems in differential geometry originate in everyday life. Consider the problem in cartography of representing the surface of the earth on a flat piece of paper. A good map should show accurately distances between any two points. Experience suggests that this is not possible on a large scale. We are all familiar with the Mercator projection which vastly distorts countries near the north and south poles. On a small scale there are fairly good maps, but are they merely approximations or can there be truly accurate maps in a mathematical sense? In other words, is there a distance-preserving bijection from an open subset of the sphere to some open subset of the plane? Such a map is an ***isometry***.

Isometry is also related to a problem in industrial design. Certain shapes such as circular cylinders and cones are easy to manufacture because they can be obtained

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from a flat sheet by bending. If we take a sheet of paper and bend it in various ways, we obtain infinitely many surfaces in space, and yet none of them appear to be a portion of a sphere or an ellipsoid. Which shapes can be obtained from one another by bending?

In 1827 Carl Friedrich Gauss laid the foundation for the differential geometry of surfaces in his work *Disquisitiones generales circa superficies curvas* (General investigation of curved surfaces). One of his great achievements was the proof of the invariance of Gaussian curvature under distance-preserving maps. This result is known as Gauss's Theorema Egregium, which means "remarkable theorem" in Latin. By the Theorema Egregium, one can use the Gaussian curvature to distinguish non-isometric surfaces. In the first eight sections of this book, our goal is to introduce enough basic constructions of differential geometry to prove the Theorema Egregium.

§1 Riemannian Manifolds

A Riemannian metric is essentially a smoothly varying inner product on the tangent space at each point of a manifold. In this section we recall some generalities about an inner product on a vector space and by means of a partition of unity argument, prove the existence of a Riemannian metric on any manifold.

1.1 Inner Products on a Vector Space

A point u in $\mathbb{R}^3$ will denote either an ordered triple (u^1, u^2, u^3) of real numbers or a column vector

$$\begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}.$$

The ***Euclidean inner product***, or the ***dot product***, on $\mathbb{R}^3$ is defined by

$$\langle u, v\rangle = \sum_{i=1}^{3} u^i v^i.$$

In terms of this, one can define the ***length*** of a vector

$$\|v\| = \sqrt{\langle v, v\rangle}, \tag{1.1}$$

the ***angle*** θ between two nonzero vectors (Figure 1.1)

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|}, \quad 0 \le \theta \le \pi, \tag{1.2}$$

and the ***arc length*** of a parametrized curve $c(t)$ in $\mathbb{R}^3$, $a \le t \le b$:

$$s = \int_a^b \|c'(t)\|\,dt.$$

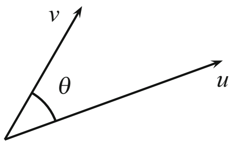


Fig. 1.1. The angle between two vectors.

Definition 1.1. An ***inner product*** on a real vector space V is a positive-definite, symmetric, bilinear form $\langle\ ,\ \rangle \colon V \times V \to \mathbb{R}$. This means that for $u, v, w \in V$ and $a, b \in \mathbb{R}$,

(i) (positive-definiteness) $\langle v, v\rangle \geq 0$; the equality holds if and only if $v = 0$.
(ii) (symmetry) $\langle u, v\rangle = \langle v, u\rangle$.
(iii) (bilinearity) $\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$.

As stated, condition (iii) is linearity in only the first argument. However, by the symmetry property (ii), condition (iii) implies linearity in the second argument as well.

Proposition 1.2 (Restriction of an inner product to a subspace). *Let $\langle\ ,\ \rangle$ be an inner product on a vector space V. If W is a subspace of V, then the restriction*

$$\langle\ ,\ \rangle_W := \langle\ ,\rangle|_{W\times W} \colon W \times W \to \mathbb{R}$$

is an inner product on W.

Proof. Problem 1.3. □

Proposition 1.3 (Nonnegative linear combination of inner products). *Let $\langle\ ,\ \rangle_i$, $i = 1, \dots, r$, be inner products on a vector V and let $a_1, \dots, a_r$ be nonnegative real numbers with at least one $a_i > 0$. Then the linear combination $\langle\ ,\ \rangle := \sum a_i \langle\ ,\ \rangle_i$ is again an inner product on V.*

Proof. Problem 1.4. □

1.2 Representations of Inner Products by Symmetric Matrices

Let $e_1, \dots, e_n$ be a basis for a vector space V. Relative to this basis we can represent vectors in V as column vectors:

$$\sum x^i e_i \longleftrightarrow \mathbf{x} = \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix}, \quad \sum y^i e_i \longleftrightarrow \mathbf{y} = \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix}.$$

By bilinearity, an inner product on V is determined completely by its values on a set of basis vectors. Let A be the $n \times n$ matrix whose entries are

$$a_{ij} = \langle e_i, e_j\rangle.$$

By the symmetry of the inner product, A is a symmetric matrix. In terms of column vectors,

$$\langle \sum x^i e_i, \sum y^j e_j \rangle = \sum a_{ij} x^i y^j = \mathbf{x}^T A \mathbf{y}.$$

Definition 1.4. An $n \times n$ symmetric matrix A is said to be ***positive-definite*** if

(i) $\mathbf{x}^T A \mathbf{x} \geq 0$ for all $\mathbf{x}$ in $\mathbb{R}^n$, and
(ii) equality holds if and only if $\mathbf{x} = \mathbf{0}$.

Thus, once a basis on V is chosen, an inner product on V determines a positive-definite symmetric matrix.

Conversely, if A is an $n \times n$ positive-definite symmetric matrix and $\{e_1, \ldots, e_n\}$ is a basis for V, then

$$\langle \sum x^i e_i, \sum y^i e_i \rangle = \sum a_{ij} x^i y^j = \mathbf{x}^T A \mathbf{y}$$

defines an inner product on V. (Problem 1.1.)

It follows that there is a one-to-one correspondence

$$\left\{ \begin{array}{l} \text{inner products on a vector} \\ \text{space } V \text{ of dimension } n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} n \times n \text{ positive-definite} \\ \text{symmetric matrices} \end{array} \right\}.$$

The dual space $V^\vee$ of a vector space V is by definition $\operatorname{Hom}(V, \mathbb{R})$, the space of all linear maps from V to $\mathbb{R}$. Let $\alpha^1, \ldots, \alpha^n$ be the basis for $V^\vee$ dual to the basis $e_1, \ldots, e_n$ for V. If $x = \sum x^i e_i \in V$, then $\alpha^i(x) = x^i$. Thus, with $x = \sum x^i e_i$, $y = \sum y^j e_j$, and $\langle e_i, e_j \rangle = a_{ij}$, one has

$$\begin{aligned} \langle x, y \rangle &= \sum a_{ij} x^i y^j = \sum a_{ij} \alpha^i(x) \alpha^j(y) \\ &= \sum a_{ij} (\alpha^i \otimes \alpha^j)(x, y). \end{aligned}$$

So in terms of the tensor product, an inner product $\langle \ , \ \rangle$ on V may be written as

$$\langle \ , \ \rangle = \sum a_{ij} \alpha^i \otimes \alpha^j,$$

where $[a_{ij}]$ is an $n \times n$ positive-definite symmetric matrix.

1.3 Riemannian Metrics

Definition 1.5. A ***Riemannian metric*** on a manifold M is the assignment to each point p in M of an inner product $\langle \ , \ \rangle_p$ on the tangent space T_pM; moreover, the assignment $p \mapsto \langle \ , \ \rangle_p$ is required to be C^∞ in the following sense: if X and Y are C^∞ vector fields on M, then $p \mapsto \langle X_p, Y_p \rangle_p$ is a C^∞ function on M. A ***Riemannian manifold*** is a pair $(M, \langle \ , \ \rangle)$ consisting of a manifold M together with a Riemannian metric $\langle \ , \ \rangle$ on M.

The ***length*** of a tangent vector $v \in T_pM$ and the angle between two tangent vectors $u, v \in T_pM$ on a Riemannian manifold are defined by the same formulas (1.1) and (1.2) as in $\mathbb{R}^3$.

Example 1.6. Since all the tangent spaces $T_p\mathbb{R}^n$ for points p in $\mathbb{R}^n$ are canonically isomorphic to $\mathbb{R}^n$, the Euclidean inner product on $\mathbb{R}^n$ gives rise to a Riemannian metric on $\mathbb{R}^n$, called the ***Euclidean metric*** on $\mathbb{R}^n$.

Example 1.7. Recall that a submanifold M of a manifold N is said to be ***regular*** if locally it is defined by the vanishing of a set of coordinates [21, Section 9]. Thus, locally a regular submanifold looks like a k-plane in $\mathbb{R}^n$. By a surface M in $\mathbb{R}^3$ we will mean a 2-dimensional regular submanifold of $\mathbb{R}^3$. At each point p in M, the tangent space T_pM is a vector subspace of $T_p\mathbb{R}^3$. The Euclidean metric on $\mathbb{R}^3$ restricts to a function

$$\langle\ ,\ \rangle_M\colon T_pM \times T_pM \to \mathbb{R},$$

which is clearly positive-definite, symmetric, and bilinear. Thus a surface in $\mathbb{R}^3$ inherits a Riemannian metric from the Euclidean metric on $\mathbb{R}^3$.

Recall that if $F\colon N \to M$ is a C^∞ map of smooth manifolds and $p \in N$ is a point in N, then the differential $F_*\colon T_pN \to T_{f(p)}M$ is the linear map of tangent spaces given by

$$(F_*X_p)g = X_p(g \circ F)$$

for any $X_p \in T_pN$ and any C^∞ function g defined on a neighborhood of $F(p)$ in M.

Definition 1.8. A C^∞ map $F\colon (N, \langle\ ,\ \rangle') \to (M, \langle\ ,\ \rangle)$ of Riemannian manifolds is said to be ***metric-preserving*** if for all $p \in N$ and tangent vectors $u, v \in T_pN$,

$$\langle u, v\rangle'_p = \langle F_*u, F_*v\rangle_{F(p)}. \tag{1.3}$$

An ***isometry*** is a metric-preserving diffeomorphism.

Example 1.9. If $F\colon N \to M$ is a diffeomorphism and $\langle\ ,\ \rangle$ is a Riemannian metric on M, then (1.3) defines an induced Riemannian metric $\langle\ ,\ \rangle'$ on N.

Example 1.10. Let N and M be the unit circle in $\mathbb{C}$. Define $F\colon N \to M$, a 2-sheeted covering space map, by $F(z) = z^2$. Give M a Riemannian metric $\langle\ ,\ \rangle$, for example, the Euclidean metric as a subspace of $\mathbb{R}^2$, and define $\langle\ ,\ \rangle'$ on N by

$$\langle v, w\rangle' = \langle F_*v, F_*w\rangle.$$

Then $\langle\ ,\ \rangle'$ is a Riemannian metric on N. The map $F\colon N \to M$ is metric-preserving but not an isometry because F is not a diffeomorphism.

Example 1.11. A torus in $\mathbb{R}^3$ inherits the Euclidean metric from $\mathbb{R}^3$. However, a torus is also the quotient space of $\mathbb{R}^2$ by the group $\mathbb{Z}^2$ acting as translations, or to put it more plainly, the quotient space of a square with the opposite edges identified (see [21, §7] for quotient spaces). In this way, it inherits a Riemannian metric from $\mathbb{R}^2$. With these two Riemannian metrics, the torus becomes two distinct Riemannian manifolds (Figure 1.2). We will show later that there is no isometry between these two Riemannian manifolds with the same underlying torus.

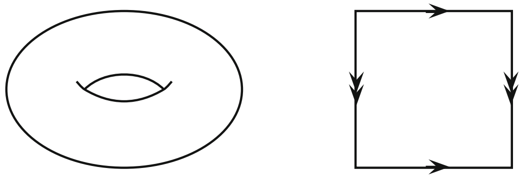


Fig. 1.2. Two Riemannian metrics on the torus.

1.4 Existence of a Riemannian Metric

A smooth manifold M is locally diffeomorphic to an open subset of a Euclidean space. The local diffeomorphism defines a Riemannian metric on a coordinate open set $(U, x^1, \ldots, x^n)$ by the same formula as for $\mathbb{R}^n$. We will write ∂_i for the coordinate vector field $\partial/\partial x^i$. If $X = \sum a^i \partial_i$ and $Y = \sum b^j \partial_j$, then the formula

$$\langle X, Y\rangle = \sum a^i b^i \tag{1.4}$$

defines a Riemannian metric on U.

To construct a Riemannian metric on M one needs to piece together the Riemannian metrics on the various coordinate open sets of an atlas. The standard tool for this is the partition of unity, whose definition we recall now. A collection $\{S_\alpha\}$ of subsets of a topological space S is said to be ***locally finite*** if every point $p \in S$ has a neighborhood U_p that intersects only finitely many of the subsets S_α. The ***support*** of a function $f\colon S \to \mathbb{R}$ is the closure of the subset of S on which $f \neq 0$:

$$\operatorname{supp} f = \operatorname{cl}\{x \in S \mid f(x) \neq 0\}.$$

Suppose $\{U_\alpha\}_{\alpha \in \mathrm{A}}$ is an open cover of a manifold M. A collection of nonnegative C^∞ functions

$$\rho_\alpha\colon M \to \mathbb{R}, \quad \alpha \in \mathrm{A},$$

is called a C^∞ ***partition of unity subordinate to*** $\{U_\alpha\}$ if

(i) $\operatorname{supp} \rho_\alpha \subset U_\alpha$ for all α,
(ii) the collection of supports, $\{\operatorname{supp} \rho_\alpha\}_{\alpha \in \mathrm{A}}$, is locally finite,
(iii) $\sum_{\alpha \in \mathrm{A}} \rho_\alpha = 1$.

The local finiteness of the supports guarantees that every point p has a neighborhood U_p over which the sum in (iii) is a finite sum. (For the existence of a C^∞ partition of unity, see [21, Appendix C].)

Theorem 1.12. *On every manifold M there is a Riemannian metric.*

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on M. Using the coordinates on U_α, we define as in (1.4) a Riemannian metric $\langle\ ,\ \rangle_\alpha$ on U_α. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. By the local finiteness of the collection $\{\operatorname{supp} \rho_\alpha\}$, every point

p has a neighborhood U_p on which only finitely many of the ρ_α's are nonzero. Thus, $\sum \rho_\alpha \langle\ ,\ \rangle_\alpha$ is a finite sum on U_p. By Proposition 1.3, at each point p the sum $\sum \rho_\alpha \langle\ ,\ \rangle_\alpha$ is an inner product on T_pM.

To show that $\sum \rho_\alpha \langle\ ,\ \rangle_\alpha$ is C^∞, let X and Y be C^∞ vector fields on M. Since $\sum \rho_\alpha \langle X, Y\rangle_\alpha$ is a finite sum of C^∞ functions on U_p, it is C^∞ on U_p. Since p was arbitrary, $\sum \rho_\alpha \langle X, Y\rangle_\alpha$ is C^∞ on M. □

Problems

1.1.* Positive-definite symmetric matrix
Show that if A is an $n \times n$ positive-definite symmetric matrix and $\{e_1, \dots, e_n\}$ is a basis for V, then

$$\left\langle \sum x^i e_i, \sum y^i e_i \right\rangle = \sum a_{ij} x^i y^j = \mathbf{x}^T A \mathbf{y}$$

defines an inner product on V.

1.2.* Inner product
Let V be an inner product space with inner product $\langle\ ,\ \rangle$. For u, v in V, prove that $\langle u, w\rangle = \langle v, w\rangle$ for all w in V if and only if $u = v$.

1.3. Restriction of an inner product to a subspace
Prove Proposition 1.2.

1.4.* Positive linear combination of inner products
Prove Proposition 1.3.

1.5.* Extending a vector to a vector field
Let M be a manifold. Show that for any tangent vector $v \in T_pM$, there is a C^∞ vector field X on M such that $X_p = v$.

1.6.* Equality of vector fields
Suppose $(M, \langle\ ,\ \rangle)$ is a Riemannian manifold. Show that two C^∞ vector fields $X, Y \in \mathfrak{X}(M)$ are equal if and only if $\langle X, Z\rangle = \langle Y, Z\rangle$ for all C^∞ vector fields $Z \in \mathfrak{X}(M)$.

1.7.* Upper half-plane
Let

$$\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}.$$

At each point $p = (x,y) \in \mathbb{H}^2$, define

$$\langle\ ,\ \rangle_{\mathbb{H}^2} : T_p\mathbb{H}^2 \times T_p\mathbb{H}^2 \to \mathbb{R}$$

by

$$\langle u, v\rangle_{\mathbb{H}^2} = \frac{1}{y^2}\langle u, v\rangle,$$

where $\langle\ ,\ \rangle$ is the usual Euclidean inner product. Show that $\langle\ ,\ \rangle_{\mathbb{H}^2}$ is a Riemannian metric on $\mathbb{H}^2$.

1.8. Product rule in $\mathbb{R}^n$

If $f, g\colon \mathbb{R} \to \mathbb{R}^n$ are differentiable vector-valued functions, show that $\langle f, g\rangle\colon \mathbb{R} \to \mathbb{R}$ is differentiable and

$$\langle f, g\rangle' = \langle f', g\rangle + \langle f, g'\rangle.$$

(Here f' means df/dt.)

1.9. Product rule in an inner product space

An inner product space $(V, \langle\ ,\ \rangle)$ is automatically a normed vector space, with norm $\|v\| = \sqrt{\langle v, v\rangle}$. The ***derivative*** of a function $f\colon \mathbb{R} \to V$ is defined to be

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h},$$

provided that the limit exists, where the limit is taken with respect to the norm $\|\ \|$. If $f, g\colon \mathbb{R} \to V$ are differentiable functions, show that $\langle f, g\rangle\colon \mathbb{R} \to \mathbb{R}$ is differentiable and

$$\langle f, g\rangle' = \langle f', g\rangle + \langle f, g'\rangle.$$

§2 Curves

In common usage a ***curve*** in a manifold M can mean two things. Either it is a ***parametrized curve***, i.e., a smooth map $c\colon [a,b] \to M$, or it is the set of points in M that is the image of this map. By definition, a smooth map on a closed interval is a smooth map on some open set containing the interval. For us, a curve will always mean a parametrized curve. When the occasion calls for it, we will refer to the image of a parametrized curve as a ***geometric curve***.

A regular curve is a parametrized curve whose velocity is never zero. A regular curve can be reparametrized by arc length. In this section we define the signed curvature of a regular plane curve in terms of the second derivative of its arc length parametrization.

2.1 Regular Curves

Definition 2.1. A parametrized curve $c\colon [a,b] \to M$ is ***regular*** if its velocity $c'(t)$ is never zero for all t in the domain $[a,b]$. In other words, a regular curve in M is an immersion: $[a,b] \to M$.

Example 2.2. The curve $c\colon [-1,1] \to \mathbb{R}^2$,

$$c(t) = (t^3, t^2),$$

is not regular at $t = 0$ (Figure 2.1). This example shows that the image of a smooth nonregular curve need not be smooth.

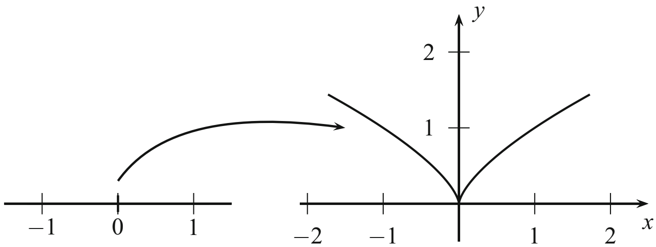


Fig. 2.1. A nonregular curve.

If $t = t(u)$ is a diffeomorphism of one closed interval with another, then $\beta(u) := c(t(u))$ is a ***reparametrization*** of the curve $c(t)$. The same geometric curve can have many different parametrizations. Among the various reparametrizations of a regular curve, the most important is the ***arc length parametrization***.

2.2 Arc Length Parametrization

As in calculus, we define the ***speed*** of a curve $c\colon [a,b] \to M$ in a Riemannian manifold M to be the magnitude $\|c'(t)\|$ of its velocity $c'(t)$, and the ***arc length*** of the curve to be

$$\ell = \int_a^b \|c'(u)\|\, du.$$

For each $t \in [a,b]$, let $s(t)$ be the arc length of the curve c restricted to $[a,t]$:

$$s(t) = \int_a^t \|c'(u)\|\, du.$$

The function $s\colon [a,b] \to [0,\ell]$ is the ***arc length function*** of the curve c. By the fundamental theorem of calculus, the derivative of s with respect to t is $s'(t) = \|c'(t)\|$, the speed of c.

Proposition 2.3. *The arc length function $s\colon [a,b] \to [0,\ell]$ of a regular curve $c\colon [a,b] \to M$ has a C^∞ inverse.*

Proof. Because $c(t)$ is regular, $s'(t) = \|c'(t)\|$ is never zero. Then $s'(t) > 0$ for all t. This implies that $s(t)$ is a monotonically increasing function, and so has an inverse $t(s)$. By the inverse function theorem, t is a C^∞ function of s. □

Thus, given a regular curve $c(t)$, we can write t as a C^∞ function of the arc length s to get the ***arc length parametrization*** $\gamma(s) = c\big(t(s)\big)$.

Proposition 2.4. *A curve is parametrized by arc length if and only if it has unit speed and its parameter starts at 0.*

Proof. As noted above, the speed of a curve $c\colon [a,b] \to M$ can be computed as the rate of change of the arc length s with respect to $t \in [a,b]$:

$$\|c'(t)\| = \frac{ds}{dt}.$$

Let $\gamma(s)$ be the arc length reparametrization of c. Since $s(a) = 0$, the parameter s starts at 0. By the chain rule, the velocity of γ is

$$\gamma'(s) = c'(t(s))t'(s).$$

Hence, the speed of γ is

$$\|\gamma'(s)\| = \|c'(t(s))\|\; |t'(s)| = \frac{ds}{dt}\left|\frac{dt}{ds}\right| = \left|\frac{ds}{dt}\frac{dt}{ds}\right| = 1.$$

Conversely, if a curve $c(t)$ has unit speed, then its arc length is

$$s(t) = \int_a^t \|c'(u)\|\, du = \int_a^t 1\, du = t - a.$$

If $a = 0$, then $s = t$. So a unit-speed curve starting at $t = 0$ is parametrized by arc length. □

Example 2.5. The curve $c\colon [0,2\pi] \to \mathbb{R}^2$,

$$c(t) = (a\cos t, a\sin t), \qquad a > 0,$$

is regular. Its image is the circle of radius a centered at the origin. Its arc length function is

$$\begin{aligned} s &= \int_0^t \|c'(u)\|\,du = \int_0^t \left\| \begin{bmatrix} -a\sin u \\ a\cos u \end{bmatrix} \right\| du \\ &= \int_0^t a\,du = at. \end{aligned}$$

Hence, $t = s/a$ and the arc length parametrization is

$$\gamma(s) = \left(a\cos\frac{s}{a}, a\sin\frac{s}{a}\right).$$

2.3 Signed Curvature of a Plane Curve

We all have an intuitive idea of what curvature means. For example, a small circle appears to curve more than a large circle. In this section, we will quantify the notion of curvature for a curve in the plane $\mathbb{R}^2$.

Our plane curve $\gamma\colon [0,\ell] \to \mathbb{R}^2$ will be parametrized by the arc length s. Then the velocity vector $T(s) = \gamma'(s)$ has unit length and is tangent to the curve at the point $p = \gamma(s)$. A reasonable measure of curvature at p is the magnitude of the derivative

$$T'(s) = \frac{dT}{ds}(s) = \gamma''(s),$$

since the faster T changes, the more the curve bends. However, in order to distinguish the directions in which the curve can bend, we will define a curvature with a sign.

There are two unit vectors in the plane perpendicular to $T(s)$ at p. We can choose either one to be $\mathbf{n}(s)$, but usually $\mathbf{n}(s)$ is chosen so that the pair $(T(s), \mathbf{n}(s))$ is oriented positively in the plane, i.e., counterclockwise.

Denote by $\langle\ ,\ \rangle$ the Euclidean inner product on $\mathbb{R}^2$. Since T has unit length,

$$\langle T, T\rangle = 1.$$

Using the product rule (Problem 1.8) to differentiate this equation with respect to s gives

$$\langle T', T\rangle + \langle T, T'\rangle = 0,$$

or

$$2\langle T', T\rangle = 0.$$

Thus, T' is perpendicular to T and so it must be a multiple of $\mathbf{n}$. The scalar κ such that

$$T' = \kappa\mathbf{n}$$

is called the ***signed curvature***, or simply the ***curvature***, of the plane curve at $p = \gamma(s)$. We can also write

$$\kappa = \langle T', \mathbf{n} \rangle = \langle \gamma'', \mathbf{n} \rangle.$$

The sign of the curvature depends on the choice of $\mathbf{n}$; it indicates whether the curve is bending towards $\mathbf{n}$ or away from $\mathbf{n}$ (Figure 2.2).

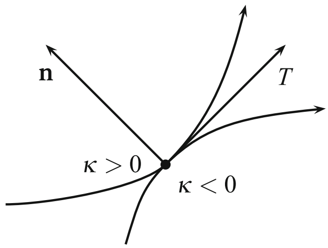


Fig. 2.2. The sign of the curvature κ.

Example 2.6 (*The circle*). By Example 2.5, the circle of radius a centered at the origin has arc length parametrization

$$\gamma(s) = \left(a\cos\frac{s}{a}, a\sin\frac{s}{a} \right), \quad 0 \le s \le 2\pi a.$$

The unit tangent vector to the curve is

$$T(s) = \gamma'(s) = \begin{bmatrix} -\sin\frac{s}{a} \\ \cos\frac{s}{a} \end{bmatrix}.$$

Its derivative is

$$T'(s) = \gamma''(s) = \begin{bmatrix} -\frac{1}{a}\cos\frac{s}{a} \\ -\frac{1}{a}\sin\frac{s}{a} \end{bmatrix}.$$

We choose the unit normal $\mathbf{n}$ so that the pair $(T, \mathbf{n})$ is oriented counterclockwise. This means $\mathbf{n}$ is obtained from T by multiplying by the rotation matrix

$$\operatorname{rot}\left(\frac{\pi}{2}\right) = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Hence,

$$\mathbf{n} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} T = \begin{bmatrix} -\cos\frac{s}{a} \\ -\sin\frac{s}{a} \end{bmatrix}.$$

So $T' = (1/a)\mathbf{n}$, which shows that the signed curvature $\kappa(s)$ of a circle of radius a is $1/a$, independent of s. This accords with the intuition that the larger the radius of a circle, the smaller the curvature.

2.4 Orientation and Curvature

A curve whose endpoints are fixed has two possible arc length parametrizations, depending on how the curve is oriented. If the arc length of the curve is ℓ, then the two parametrizations $\gamma(s)$, $\tilde{\gamma}(s)$ are related by

$$\tilde{\gamma}(s) = \gamma(\ell - s).$$

Differentiating with respect to the arc length s gives

$$\tilde{T}(s) := \tilde{\gamma}'(s) = -\gamma'(\ell - s) = -T(\ell - s) \quad \text{and} \quad \tilde{\gamma}''(s) = \gamma''(\ell - s).$$

Rotating the tangent vector $\tilde{T}(s)$ by 90° amounts to multiplying $\tilde{T}(s)$ on the left by the rotation matrix $\operatorname{rot}(\pi/2)$. Thus,

$$\tilde{\mathbf{n}}(s) = \operatorname{rot}\left(\frac{\pi}{2}\right)\tilde{T}(s) = -\operatorname{rot}\left(\frac{\pi}{2}\right)T(\ell - s) = -\mathbf{n}(\ell - s).$$

It follows that the signed curvature of $\tilde{\gamma}$ at $\tilde{\gamma}(s) = \gamma(\ell - s)$ is

$$\tilde{\kappa}(s) = \langle \tilde{\gamma}''(s), \tilde{\mathbf{n}}(s)\rangle = \langle \gamma''(\ell - s), -\mathbf{n}(\ell - s)\rangle = -\kappa(\ell - s).$$

In summary, reversing the orientation of a plane curve reverses the sign of its signed curvature at any point.

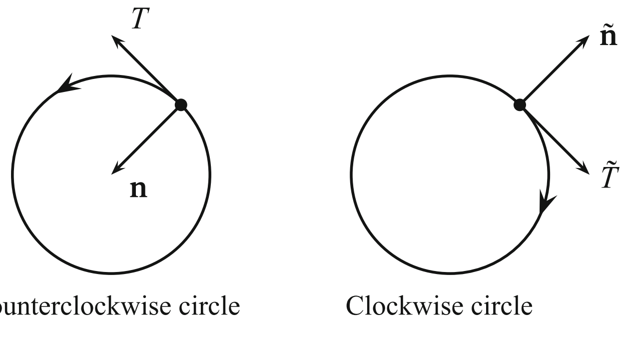


Fig. 2.3. Reversing the orientation of a curve changes the sign of the curvature at a point.

Example 2.7. The counterclockwise circle

$$\gamma(s) = (a\cos s/a, a\sin s/a), \quad 0 \le s \le 2\pi a,$$

has signed curvature $1/a$, while the clockwise circle

$$\tilde{\gamma}(s) = \gamma(-s), \quad 0 \le s \le 2\pi a,$$

has signed curvature $-1/a$. Geometrically, the unit tangent vector T of the counterclockwise circle turns towards the normal vector $\mathbf{n}$, while the unit tangent vector $\tilde{T}$ of the clockwise circle turns away from the normal vector $\tilde{\mathbf{n}}$ (Figure 2.3).

Problems

2.1.* Signed curvature
Let $T(s)$ be the unit tangent vector field on a plane curve $\gamma(s)$ parametrized by arc length. Write

$$T(s) = \begin{bmatrix} \cos\theta(s) \\ \sin\theta(s) \end{bmatrix},$$

where $\theta(s)$ is the angle of $T(s)$ with respect to the positive horizontal axis. Show that the signed curvature κ is the derivative $d\theta/ds$.

2.2. Curvature of a unit-speed plane curve
Suppose $\gamma(s) = (x(s), y(s))$ is a unit-speed curve in the plane. Write $x' = x'(s) = dx/ds$ and $y' = y'(s) = dy/ds$.

(a) Show that the signed curvature κ of $\gamma(s)$ is $-x''/y'$ when $y' \neq 0$ and y''/x' when $x' \neq 0$.
(b) For a unit-speed curve, $\theta = \tan^{-1}(y'/x')$. By calculating $d\theta/ds$, show that the signed curvature is $\kappa = x'y'' - x''y'$. (If $x' = 0$, then because the curve is unit speed, $y' \neq 0$ and $\theta = \tan^{-1}(+\infty) = \pi/2$.)

2.3. Curvature of a regular plane curve
In physics literature, it is customary to denote the derivative with respect to time by a dot, e.g., $\dot{x} = dx/dt$, and the derivative with respect to distance by a prime, e.g., $x' = dx/ds$. We will sometimes follow this convention. Let $c(t) = (x(t), y(t))$ be a regular plane curve. Using Problem 2.1 and the chain rule $d\theta/ds = (d\theta/dt)/(ds/dt)$, show that the signed curvature of the curve $c(t)$ is

$$\kappa = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}},$$

where $\dot{x} = dx/dt$ and $\ddot{x} = d^2x/dt^2$.

2.4. Curvature of a graph in the plane
The ***graph*** of a C^∞ function $y = f(x)$ is the set

$$\{(x, f(x)) \mid x \in \mathbb{R}\}$$

in the plane. Showed that the signed curvature of this graph at $(x, f(x))$ is

$$\kappa = \frac{f''(x)}{\left(1 + (f'(x))\right)^{3/2}}.$$

2.5. Curvature of an ellipse
The ellipse with equation $x^2/a^2 + y^2/b^2 = 1$ in the (x,y)-plane (Figure 2.4) can be parametrized by

$$x = a\cos t, \quad y = b\sin t, \qquad 0 \le t \le 2\pi.$$

Find the curvature of the ellipse at an arbitrary point (x, y).

2.6. Arc length of a cuspidal cubic
The cuspidal cubic $x^2 = y^3$ (Figure 2.4) can be parametrized by (t^3, t^2). Find its arc length from $t = 0$ to $t = a$.

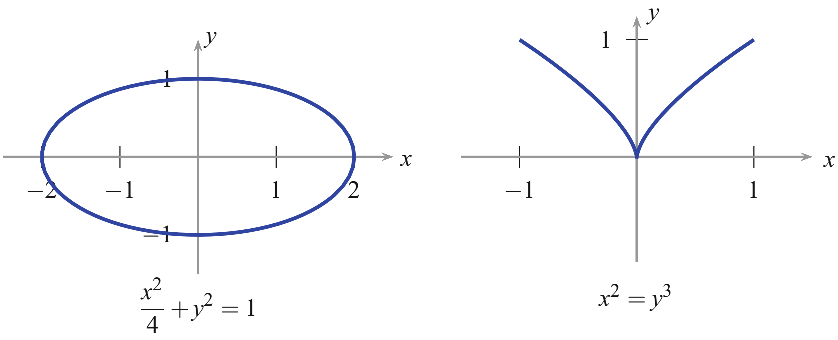


Fig. 2.4. An ellipse and a cuspidal cubic

2.7.* Curvature of a space curve
Let I be a closed interval in $\mathbb{R}$. If $\gamma\colon I \to \mathbb{R}^3$ is a regular space curve parametrized by arc length, its ***curvature*** κ at $\gamma(s)$ is defined to be $\|\gamma''(s)\|$.[1] Consider the helix $c(t) = (a\cos t, a\sin t, bt)$ in space.

(a) Reparametrize c by arc length: $\gamma(s) = c(t(s))$.
(b) Compute the curvature of the helix at $\gamma(s)$.

2.8. Curvature of a line in space
Show that the curvature of a line in space is zero at every point.

2.9. The Frenet–Serret formulas
Let $\gamma\colon I \to \mathbb{R}^3$ be a regular space curve parametrized by arc length. Then $T = \gamma'(s)$ is tangent to γ at $\gamma(s)$ and has unit length. Assume that $\gamma''(s) \neq 0$.

(a) Prove that $\gamma''(s)$ is normal to T.
(b) Let N be the unit vector $\gamma''(s)/\|\gamma''(s)\|$. Then $T' = \kappa N$, where κ is the curvature of the space curve.
(c) The unit vector $B = T \times N$ is called the ***binormal*** of γ at $\gamma(s)$. The three vectors T, N, B form an orthonormal basis for $\mathbb{R}^3$ at $\gamma(s)$, called the ***Frenet–Serret frame*** (Figure 2.5). Prove that
$$N' = -\kappa T + \tau B$$
for some real number τ, which is called the ***torsion*** of the unit-speed curve γ.
(d) Prove that $B' = -\tau N$.

The set of equations

$$\begin{aligned} T' &= \qquad\quad \kappa N \\ N' &= -\kappa T \qquad\quad + \tau B \\ B' &= \qquad\quad -\tau N \end{aligned}$$

[1]The curvature of a space curve is always nonnegative, while the signed curvature of a plane curve could be negative. To distinguish the two for a space curve that happens to be a plane curve, one can use κ_2 for the signed curvature. We will use the same notation κ for both, as it is clear from the context what is meant.

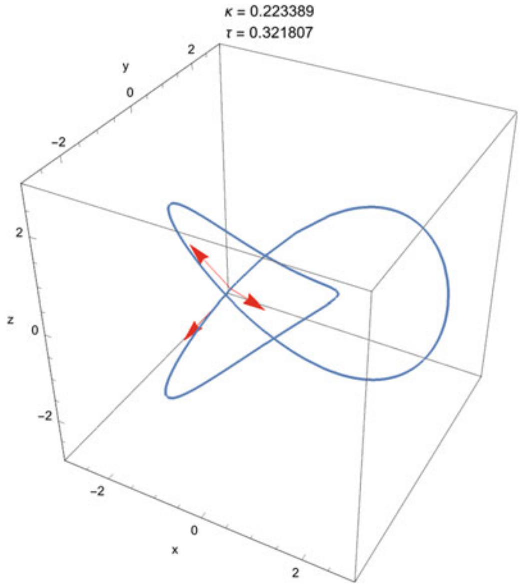


Fig. 2.5. A Frenet–Serret frame.

is called the ***Frenet–Serret formulas***. In matrix notation,

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} & \kappa & \\ -\kappa & & \tau \\ & -\tau & \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$



Jean Frédéric Frenet (1816–1900) and Joseph Alfred Serret (1819–1885)

§3 Surfaces in Space

There are several ways to generalize the curvature of a plane curve to a surface. One way, following Euler (1760), is to consider the curvature of all the normal sections of the surface at a point. A second way is to study the derivative of a unit normal vector field on the surface. In this section we use Euler's method to define several measurements of curvature at a point on a surface. We then state the two theorems, Gauss's Theorema Egregium and the Gauss–Bonnet theorem, that will serve as guideposts in our study of differential geometry. We will take up the relationship between curvature and the derivative of a normal vector field in Section 5.

3.1 Principal, Mean, and Gaussian Curvatures

Recall that a ***regular submanifold*** of a manifold $\tilde{M}$ is a subset of the manifold $\tilde{M}$ locally defined by the vanishing of coordinate functions (see [21, Section 9] for a discussion of regular submanifolds). By a surface in $\mathbb{R}^3$, we mean a 2-dimensional regular submanifold of $\mathbb{R}^3$. Let p be a point on a surface M in $\mathbb{R}^3$. A ***normal vector*** to M at p is a vector $N_p \in T_p\mathbb{R}^3$ that is orthogonal to the tangent plane T_pM. A ***normal vector field*** on M is a function N that assigns to each $p \in M$ a normal vector N_p at p. If N is a normal vector field on M, then at each point $p \in M$, we can write

$$N_p = \sum_{i=1}^{3} a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p .$$

The normal vector field N on M is said to be C^∞ if the coefficient functions a^1, a^2, a^3 are C^∞ functions on M.

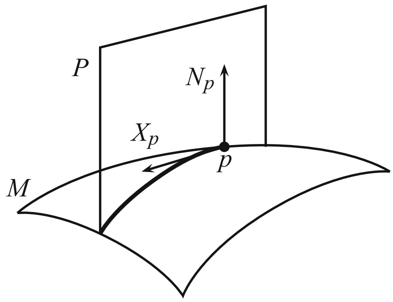


Fig. 3.1. Normal section at p.

Let N be a C^∞ unit normal vector field on a neighborhood of p in M. Denote by N_p the unit normal vector at p. Under the canonical identification of $T_p\mathbb{R}^3$ with $\mathbb{R}^3$, every plane P through N_p slices the surface M along a plane curve $P \cap M$ through p.

By the transversality theorem from differential topology, the intersection $P \cap M$, being transversal, is smooth (see Problem 3.2). We call such a plane curve a ***normal section*** of the surface through p (Figure 3.1). Assuming that the normal sections have C^∞ parametrizations, which we will show later, we can compute the curvature of a normal section with respect to N_p. The collection of the curvatures at p of all the normal sections gives a fairly good picture of how the surface curves at p.

More precisely, each unit tangent vector X_p to the surface M at p determines together with N_p a plane that slices M along a normal section. Moreover, the unit tangent vector X_p determines an orientation of the normal section. Let $\gamma(s)$ be the arc length parametrization of this normal section with initial point $\gamma(0) = p$ and initial vector $\gamma'(0) = X_p$. Note that $\gamma(s)$ is completely determined by the unit tangent vector X_p. Define the ***normal curvature*** of the normal section $\gamma(s)$ at p with respect to N_p by

$$\kappa(X_p) = \langle \gamma''(0), N_p \rangle. \tag{3.1}$$

Of course, this N_p is not always the same as the $\mathbf{n}(0)$ in Section 2, which was obtained by rotating the unit tangent vector 90° counterclockwise in $\mathbb{R}^2$; an arbitrary plane in $\mathbb{R}^3$ does not have a preferred orientation.



Leonhard Euler

(1707–1783)

(Portrait by Jakob Emanuel Handmann, 1753)

Since the set of all unit vectors in T_pM is a circle, we have a function

$$\kappa \colon S^1 \to \mathbb{R}.$$

Clearly, $\kappa(-X_p) = \kappa(X_p)$ for $X_p \in S^1$, because replacing a unit tangent vector by its negative simply reverses the orientation of the normal section, which reverses the sign of the first derivative $\gamma'(s)$ but does not change the sign of the second derivative $\gamma''(s)$.

The maximum and minimum values κ_1, κ_2 of the function κ are the ***principal curvatures*** of the surface at p. Their average $(\kappa_1 + \kappa_2)/2$ is the ***mean curvature*** H, and their product $\kappa_1 \kappa_2$ the ***Gaussian curvature*** K. A unit direction $X_p \in T_pM$ along which a principal curvature occurs is called a ***principal direction***. Note that if X_p is a principal direction, then so is $-X_p$, since $\kappa(-X_p) = \kappa(X_p)$. If κ_1 and κ_2 are equal, then every unit vector in T_pM is a principal direction.

Remark 3.1. Using $-N_p$ instead of N_p reverses the signs of all the normal curvatures at p, as one sees from (3.1). This will change the sign of the mean curvature, but it leaves invariant the Gaussian curvature. Thus, the Gaussian curvature K is independent of the choice of the unit normal vector field N.

Example 3.2 (*Sphere of radius a*). Every normal section of a sphere of radius a is a circle of radius a. With respect to the unit inward-pointing unit normal vector field, the principal curvatures are both $1/a$ (see Example 2.6), the mean curvature is $H = 1/a$ and the Gaussian curvature is $K = 1/a^2$.

Example 3.3. For a plane M in $\mathbb{R}^3$ the principal curvatures, mean curvature, and Gaussian curvature are all zero.

Example 3.4. For a cylinder of radius a with a unit inward normal, it appears that the principal curvatures are 0 and $1/a$, corresponding to normal sections that are a line and a circle, respectively (Figure 3.2). We will establish this rigorously in Section 5. Hence, the mean curvature is $1/2a$ and the Gaussian curvature is 0. If we use the unit outward normal on the cylinder, then the principal curvatures are 0 and $-1/a$, and the mean curvature is $-1/2a$, but the Gaussian curvature is still 0.

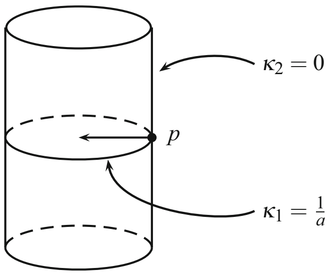


Fig. 3.2. Principal curvatures of a cylinder.

3.2 Gauss's Theorema Egregium

Since the plane is locally isometric to a cylinder, Examples 3.3 and 3.4 show that the principal curvatures κ_1, κ_2 and the mean curvature H are not isometric invariants. It is an astonishing fact that although neither κ_1 nor κ_2 is invariant under isometries, their product, the Gaussian curvature K, is. This is the content of the Theorema Egregium of Gauss.

Another way to appreciate the significance of this theorem is to think in terms of isometric embeddings. We may think of portions of the plane and the cylinder as different isometric embeddings of the same planar region. The examples show that the principal curvatures κ_1, κ_2 and the mean curvature depend on the embedding. The product $\kappa_1\kappa_2$ would ***a priori*** seem to depend on the embedding, but in fact does not. In the next few sections we will develop the machinery to prove the Theorema Egregium.



Carl Friedrich Gauss

(1777–1855)

(Artist: Gottlieb Biermann, 1887)

3.3 The Gauss–Bonnet Theorem

For an oriented surface M in $\mathbb{R}^3$ the Gaussian curvature K is a function on the surface. If the surface is compact, we can integrate K to obtain a single number $\int_M K\,dS$. Here the integral is the usual surface integral from vector calculus.

Example 3.5. For the sphere M of radius a (Figure 3.3), the integral of the Gaussian curvature is

$$\begin{aligned}\int_M K\,dS &= \int_M \frac{1}{a^2}\,dS = \frac{1}{a^2}\int_M 1\,dS \\ &= \frac{1}{a^2}\,(\text{surface area of } M) \\ &= \frac{1}{a^2}4\pi a^2 = 4\pi.\end{aligned}$$

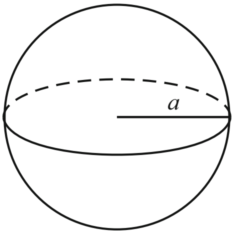


Fig. 3.3. Sphere of radius a.

Notice that although the Gaussian curvature of a sphere depends on the radius, in the integration the radius cancels out and the final answer is independent of the radius.

Example 3.5 is a special case of the Gauss–Bonnet theorem, which for a compact oriented surface M in $\mathbb{R}^3$ asserts that

$$\int_M K\,dS = 2\pi\chi(M), \tag{3.2}$$

where $\chi(M)$ denotes the Euler characteristic. Equation (3.2) is a rather unexpected formula, for on the left-hand side the Gaussian curvature K depends on a notion of distance, but the right-hand side is a topological invariant, independent of any Riemannian metric. Somehow, in the integration process, all the metric information gets canceled out, leaving us with a topological invariant. For the 2-sphere, the Euler characteristic is 2 and the theorem checks with the computation in Example 3.5.



Pierre Ossian Bonnet

(1819–1892)

In due course we will study the theory of characteristic classes for vector bundles, a vast generalization of the Gauss–Bonnet theorem.

Problems

3.1. Principal curvatures in terms of K and H
Compute the principal curvatures κ_1, κ_2 at a point of an oriented surface in $\mathbb{R}^3$ in terms of its Gaussian curvature K and mean curvature H.

3.2. Normal section at a point
Suppose M is a smooth surface in $\mathbb{R}^3$, p a point in M, and N a smooth unit normal vector field on a neighborhood of p in M. Let P be the plane spanned by a unit tangent vector $X_p \in T_pM$ and the unit normal vector N_p. Denote by $C := P \cap M$ the normal section of the surface M at p cut out by the plane P.

(a) The plane P is the zero set of some linear function $f(x,y,z)$. Let $\bar{f}\colon M \to \mathbb{R}$ be the restriction of f to M. Then the normal section C is precisely the level set $\bar{f}^{-1}(0)$. Show that p is a regular point of $\bar{f}$, i.e., that the differential $\bar{f}_{*,p}\colon T_pM \to T_0\mathbb{R}$ is surjective. (*Hint*: Which map is $f_{*,p}\colon \mathbb{R}^3 = T_p\mathbb{R}^3 \to T_{f(p)}\mathbb{R} = \mathbb{R}$? What is its kernel?)
(b) Show that a normal section of M at p is a regular submanifold of dimension one in a neighborhood of p. (*Hint*: Apply [21, Proposition 11.4] and the regular level set theorem [21, Theorem 9.9].)

3.3. Regular submanifold of dimension one
Show that if a curve C in a smooth surface is a regular submanifold of dimension one in a neighborhood of a point $p \in C$, then C has a C^∞ parametrization near p. (*Hint*: Relative to an adapted chart (U, x^1, x^2) centered at p, the curve C is the x^1-axis. A C^∞ parametrization of the x^1-axis in the (x^1, x^2)-plane is $(x^1, x^2) = (t, 0)$.)

§4 Directional Derivatives in Euclidean Space

The directional derivative is one way of differentiating vector fields on $\mathbb{R}^n$ with respect to a tangent vector. In this section we extend the calculus definition of the directional derivative of a function to the directional derivative of a vector field along a submanifold of $\mathbb{R}^n$.

4.1 Directional Derivatives in Euclidean Space

Suppose $X_p = \sum a^i \partial/\partial x^i|_p$ is a tangent vector at a point $p = (p^1, \dots, p^n)$ in $\mathbb{R}^n$ and $f(x^1, \dots, x^n)$ is a C^∞ function in a neighborhood of p in $\mathbb{R}^n$. To compute the ***directional derivative*** of f at p in the direction X_p, we first write down a set of parametric equations for the line through p in the direction X_p:

$$x^i = p^i + ta^i, \qquad i = 1, \dots, n.$$

Let $a = (a^1, \dots, a^n)$. Then the directional derivative $D_{X_p} f$ is

$$\begin{aligned} D_{X_p} f &= \lim_{t\to 0} \frac{f(p+ta) - f(p)}{t} = \left.\frac{d}{dt}\right|_{t=0} f(p+ta) \qquad (4.1) \\ &= \sum \left.\frac{\partial f}{\partial x^i}\right|_p \cdot \left.\frac{dx^i}{dt}\right|_0 = \sum \left.\frac{\partial f}{\partial x^i}\right|_p \cdot a^i \qquad \text{(by the chain rule)} \\ &= \left(\sum a^i \left.\frac{\partial}{\partial x^i}\right|_p\right) f = X_p f. \end{aligned}$$

In calculus, X_p is required to be a unit vector, but we will allow X_p to be an arbitrary vector at p.

As a shorthand, we write ∂_i for $\partial/\partial x^i$. The ***directional derivative*** at p of a C^∞ vector field $Y = \sum b^i \partial_i = \sum b^i\, \partial/\partial x^i$ on $\mathbb{R}^n$ in the direction X_p is defined to be

$$D_{X_p} Y = \sum (X_p b^i) \left.\frac{\partial}{\partial x^i}\right|_p. \qquad (4.2)$$

This formula shows clearly that $D_{X_p} Y$ is $\mathbb{R}$-linear in X_p.

Although (4.1) computes the directional derivative using the values of f along a line through p, we can in fact use any curve $c(t)$ with initial point p and initial vector X_p (Figure 4.1), for

$$D_{X_p} f = X_p f = c'(0) f = c_* \left(\left.\frac{d}{dt}\right|_{t=0}\right) f = \left.\frac{d}{dt}\right|_{t=0} f(c(t)).$$

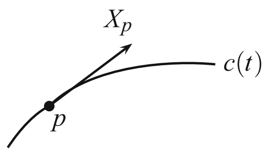


Fig. 4.1. Tangent vector at a point

Remark 4.1. Thus, for $D_{X_p} f$ to be defined, it is not necessary that f be defined in an open neighborhood of p. As long as f is defined along some curve starting at p with initial velocity X_p, the directional derivative $D_{X_p} f$ will make sense. A similar remark applies to the directional derivative $D_{X_p} Y$ of a vector field Y.

When X is a C^∞ vector field on $\mathbb{R}^n$, not just a vector at p, we define the vector field $D_X Y$ on $\mathbb{R}^n$ by

$$(D_X Y)_p = D_{X_p} Y \quad \text{for all } p \in \mathbb{R}^n.$$

Equation (4.2) shows that if X and Y are C^∞ vector fields on $\mathbb{R}^n$, then so is $D_X Y$. Let $\mathfrak{X}(\mathbb{R}^n)$ be the vector space of all C^∞ vector fields on $\mathbb{R}^n$. The directional derivative in $\mathbb{R}^n$ gives a map

$$D\colon \mathfrak{X}(\mathbb{R}^n) \times \mathfrak{X}(\mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n),$$

which we write as $D_X Y$ instead of $D(X,Y)$. Let $\mathcal{F} = C^\infty(\mathbb{R}^n)$ be the ring of C^∞ functions on $\mathbb{R}^n$. Then $\mathfrak{X}(\mathbb{R}^n)$ is both a vector space over $\mathbb{R}$ and a module over $\mathcal{F}$.

Proposition 4.2. *For $X, Y \in \mathfrak{X}(\mathbb{R}^n)$, the directional derivative $D_X Y$ satisfies the following properties:*

(i) *$D_X Y$ is $\mathcal{F}$-linear in X and $\mathbb{R}$-linear in Y;*
(ii) (Leibniz rule) *if f is a C^∞ function on $\mathbb{R}^n$, then*

$$D_X(fY) = (Xf)Y + fD_X Y.$$

Proof. (i) Let f be a C^∞ function on $\mathbb{R}^n$ and p an arbitrary point of $\mathbb{R}^n$. Then

$$\begin{aligned}
(D_{fX} Y)_p &= D_{f(p)X_p} Y \\
&= f(p) D_{X_p} Y \qquad \text{(because } D_{X_p} Y \text{ is } \mathbb{R}\text{-linear in } X_p \text{ by (4.2))} \\
&= (fD_X Y)_p.
\end{aligned}$$

For $Z \in \mathfrak{X}(\mathbb{R}^n)$, we have

$$(D_{X+Z} Y)_p = D_{X_p + Z_p} Y = D_{X_p} Y + D_{Z_p} Y = (D_X Y + D_Z Y)_p.$$

This proves that $D_X Y$ is $\mathcal{F}$-linear in X. The $\mathbb{R}$-linearity in Y is clear from (4.2).
(ii) Suppose $Y = \sum b^i \partial_i$, where $b^i \in C^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned}
(D_X(fY))_p &= \sum X_p(fb^i)\, \partial_i|_p \\
&= \sum (X_p f) b^i(p)\, \partial_i|_p + \sum f(p) X_p b^i\, \partial_i|_p \\
&= (X_p f) Y_p + f(p) D_{X_p} Y \\
&= ((Xf)Y + fD_X Y)_p.
\end{aligned}$$

□

4.2 Other Properties of the Directional Derivative

Since the directional derivative D in $\mathbb{R}^n$ is an $\mathbb{R}$-bilinear map

$$D\colon \mathfrak{X}(\mathbb{R}^n) \times \mathfrak{X}(\mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n),$$

one can ask if it is symmetric, that is, for all $X, Y \in \mathfrak{X}(\mathbb{R}^n)$, is $D_X Y = D_Y X$? A simple calculation using the standard frame shows that the answer is no; in fact, if $[X,Y]$ is the Lie bracket defined in (A.2), then

$$D_X Y - D_Y X = [X,Y].$$

The quantity

$$T(X,Y) = D_X Y - D_Y X - [X,Y]$$

turns out to be fundamental in differential geometry and is called the ***torsion*** of the directional derivative D.

For each smooth vector field $X \in \mathfrak{X}(\mathbb{R}^n)$, the directional derivative $D_X : \mathfrak{X}(\mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n)$ is an $\mathbb{R}$-linear endomorphism. This gives rise to a map

$$\begin{aligned} \mathfrak{X}(\mathbb{R}^n) &\to \operatorname{End}_{\mathbb{R}}(\mathfrak{X}(\mathbb{R}^n)), \\ X &\mapsto D_X. \end{aligned} \tag{4.3}$$

The vector space $\mathfrak{X}(\mathbb{R}^n)$ of C^∞ vector fields on $\mathbb{R}^n$ is a Lie algebra under the Lie bracket of vector fields. For any vector space V, the endomorphism ring $\operatorname{End}_{\mathbb{R}}(V)$ of endomorphisms of V is also a Lie algebra, with Lie bracket

$$[A,B] = A \circ B - B \circ A, \quad A, B \in \operatorname{End}(V).$$

So the map in (4.3) is an $\mathbb{R}$-linear map of Lie algebras. It is natural to ask if it is a Lie algebra homomorphism, i.e., is

$$[D_X, D_Y] = D_{[X,Y]}?$$

The answer is yes for the directional derivative in $\mathbb{R}^n$. A measure of the deviation of the linear map $X \mapsto D_X$ from being a Lie algebra homomorphism is given by the function

$$R(X,Y) = [D_X, D_Y] - D_{[X,Y]} = D_X D_Y - D_Y D_X - D_{[X,Y]} \in \operatorname{End}_{\mathbb{R}}(\mathfrak{X}(M)),$$

called the ***curvature*** of D.

Finally, one might ask if the product rule holds for the Euclidean inner product:

$$D_Z\langle X,Y\rangle = \langle D_Z X, Y\rangle + \langle X, D_Z Y\rangle.$$

The answer is again yes. The following proposition summarizes the properties of the directional derivative in $\mathbb{R}^n$.

Proposition 4.3. *Let D be the directional derivative in $\mathbb{R}^n$ and X, Y, Z C^∞ vector fields on $\mathbb{R}^n$. Then*

(i) (zero torsion) $D_X Y - D_Y X - [X,Y] = 0$,
(ii) (zero curvature) $D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z = 0$,
(iii) (compatibility with the metric) $X\langle Y,Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$.

Proof. (i) Problem 4.2.
(ii) Let $Z = \sum c^i \partial_i \in \mathfrak{X}(\mathbb{R}^n)$. Then

$$D_X D_Y Z = D_X \left(\sum (Yc^i)\partial_i\right) = \sum (XYc^i)\partial_i.$$

By symmetry,

$$D_Y D_X Z = \sum (YXc^i)\partial_i.$$

So

$$D_X D_Y Z - D_Y D_X Z = \sum (XY - YX)c^i \partial_i = D_{[X,Y]} Z.$$

(iii) Let $Y = \sum b^i \partial_i$ and $Z = \sum c^j \partial_j \in \mathfrak{X}(\mathbb{R}^n)$. Then

$$\begin{aligned} X\langle Y,Z\rangle &= X(\sum b^i c^i) = \sum (Xb^i)c^i + \sum b^i (Xc^i), \\ &= \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle. \end{aligned}$$

□

If X and Y are smooth vector fields on a manifold M, then the *Lie derivative* $\mathcal{L}_X Y$ is another way of differentiating Y with respect to X (for a discussion of the Lie derivative, see [21, Section 20]). While the directional derivative $D_X Y$ in $\mathbb{R}^n$ is $\mathcal{F}$-linear in X, the Lie derivative $\mathcal{L}_X Y$ is not, so the two concepts are not the same. Indeed, since $\mathcal{L}_X Y = [X,Y]$, by Proposition 4.3(i), for vector fields on $\mathbb{R}^n$,

$$\mathcal{L}_X Y = D_X Y - D_Y X.$$

4.3 Vector Fields Along a Curve

Suppose $c\colon [a,b] \to M$ is a parametrized curve in a manifold M.

Definition 4.4. A ***vector field*** V ***along*** $c\colon [a,b] \to M$ is the assignment of a tangent vector $V(t) \in T_{c(t)}M$ to each $t \in [a,b]$. Such a vector field is said to be C^∞ if for every C^∞ function f on M, the function $V(t)f$ is C^∞ as a function of t.

Example 4.5. The ***velocity vector field*** $c'(t)$ of a parametrized curve c is defined by

$$c'(t) = c_{*,t}\left(\left.\frac{d}{dt}\right|_t\right) \in T_{c(t)}M.$$

It is a vector field along c.

Example 4.6. If $\tilde{V}$ is a vector field on M and $c\colon [a,b] \to M$ is a parametrized curve in M, then $\tilde{V}$ induces a vector field V along c by

$$V(t) = \tilde{V}_{c(t)}.$$

Now suppose $c\colon [a,b] \to \mathbb{R}^n$ is a curve in $\mathbb{R}^n$ and V is a C^∞ vector field along c. Then $V(t)$ can be written as a linear combination of the standard basis vectors:

$$V(t) = \sum v^i(t)\, \partial_i|_{c(t)}\,.$$

So it makes sense to differentiate V with respect to t:

$$\frac{dV}{dt}(t) = \sum \frac{dv^i}{dt}(t)\, \partial_i|_{c(t)}\,,$$

which is also a C^∞ vector field along c.

For a smooth vector field V along a curve c in an arbitrary manifold M, without further hypotheses on M, the derivative dV/dt is in general not defined. This is because an arbitrary manifold does not have a canonical frame of vector fields such as $\partial/\partial x^1, \ldots, \partial/\partial x^n$ on $\mathbb{R}^n$.

Proposition 4.7. *Let $c\colon [a,b] \to \mathbb{R}^n$ be a curve in $\mathbb{R}^n$ and let $V(t), W(t)$ be smooth vector fields along c. Then*

$$\frac{d}{dt}\langle V(t), W(t)\rangle = \left\langle \frac{dV}{dt}, W \right\rangle + \left\langle V, \frac{dW}{dt} \right\rangle.$$

Proof. Write $V(t)$ and $W(t)$ in terms of $\partial_i|_{c(t)}$ and differentiate. □

4.4 Vector Fields Along a Submanifold

Let M be a regular submanifold of a manifold $\tilde{M}$. At a point p in M, there are two kinds of tangent vectors to $\tilde{M}$, depending on whether they are tangent to M or not. For example, if M is a surface in $\mathbb{R}^3$, the vectors in a tangent vector field on M are all tangent to M, but the vectors in a normal vector field on M are not tangent to M but to $\mathbb{R}^3$.

Definition 4.8. Let M be a submanifold of a manifold $\tilde{M}$. A vector field X ***on*** M assigns to each $p \in M$ a tangent vector $X_p \in T_pM$. A vector field X ***along*** M in $\tilde{M}$ assigns to each $p \in M$ a tangent vector $X_p \in T_p\tilde{M}$. A vector field X along M is called C^∞ if for every C^∞ function f on $\tilde{M}$, the function Xf is C^∞ on M.

The distinction between these two concepts is indicated by the prepositions "on" and "along." While it may be dangerous for little prepositions to assume such grave duties, this appears to be common usage in the literature. In this terminology, a normal vector field to a surface M in $\mathbb{R}^3$ is a vector field *along* M in $\mathbb{R}^3$, but not a vector field *on* M. Of course, a vector field on M is a vector field along M.

As in Section A.3, the set of all C^∞ vector fields on a manifold M is denoted by $\mathfrak{X}(M)$. The set of all C^∞ vector fields along a submanifold M in a manifold $\tilde{M}$ will be denoted by $\Gamma(T\tilde{M}|_M)$. They are both modules over the ring $\mathcal{F} = C^\infty(M)$ of C^∞ functions on M.

4.5 Directional Derivatives on a Submanifold of $\mathbb{R}^n$

Suppose M is a regular submanifold of $\mathbb{R}^n$. At any point $p \in M$, if $X_p \in T_pM$ is a tangent vector at p and $Y = \sum b^i \partial_i$ is a vector field along M in $\mathbb{R}^n$, then the directional derivative $D_{X_p}Y$ is defined. In fact,

$$D_{X_p}Y = \sum (X_p b^i)\, \partial_i|_p .$$

As p varies over M, this allows us to associate to a C^∞ vector field X *on* M and a C^∞ vector field Y *along* M a C^∞ vector field D_XY along M by

$$D_XY = \sum (Xb^i)\partial_i \in \Gamma(T\mathbb{R}^n|_M).$$

For any $p \in M$, we have $(D_XY)_p = D_{X_p}Y$ by definition. It follows that there is an $\mathbb{R}$-bilinear map

$$D\colon \mathfrak{X}(M) \times \Gamma(T\mathbb{R}^n|_M) \to \Gamma(T\mathbb{R}^n|_M)$$
$$D(X,Y) = D_XY.$$

We call D the *directional derivative* on M. Because of the asymmetry between X and Y—one is a vector field on M, the other a vector field along M, the torsion $T(X,Y) := D_XY - D_YX - [X,Y]$ no longer makes sense. Otherwise, D satisfies the same properties as the directional derivative D on $\mathbb{R}^n$.

Proposition 4.9. *Suppose M is a regular submanifold of $\mathbb{R}^n$ and D is the directional derivative on M. For $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(T\mathbb{R}^n|_M)$,*

(i) *$D(X,Y) = D_XY$ is $\mathcal{F}$-linear in X and $\mathbb{R}$-linear in Y;*
(ii) *(Leibniz rule) if $f \in C^\infty(M)$, then*

$$D(X, fY) = D_X(fY) = (Xf)Y + fD_XY.$$

Proposition 4.10. *Suppose M is a regular submanifold of $\mathbb{R}^n$ and D is the directional derivative on M.*

(i) *(zero curvature) If $X,Y \in \mathfrak{X}(M)$ and $Z \in \Gamma(T\mathbb{R}^n|_M)$, then*

$$D_XD_YZ - D_YD_XZ - D_{[X,Y]}Z = 0.$$

(ii) *(compatibility with the metric) If $X \in \mathfrak{X}(M)$ and $Y,Z \in \Gamma(T\mathbb{R}^n|_M)$, then*

$$X\langle Y,Z\rangle = \langle D_XY, Z\rangle + \langle Y, D_XZ\rangle.$$

Proof. By Remark 4.1, these are proven in exactly the same way as Proposition 4.3. □

Suppose $\tilde{V}$ is a C^∞ vector field along a regular submanifold M in $\mathbb{R}^n$ and $c\colon [a,b] \to M$ is a C^∞ curve in M. Then $\tilde{V}$ induces a vector field V along c:

$$V(t) = \tilde{V}_{c(t)} \in T_{c(t)}\mathbb{R}^n.$$

Proposition 4.11. *Differentiation with respect to t of a vector field along a curve is the directional derivative in the tangent direction:*

$$\frac{dV}{dt} = D_{c'(t)}\tilde{V}.$$

Proof. Let

$$V(t) = \sum v^i(t)\, \partial_i|_{c(t)} \quad \text{and} \quad \tilde{V}_p = \sum \tilde{v}^i(p)\, \partial_i|_p \text{ for } p \in M.$$

Since $V(t) = \tilde{V}_{c(t)}$,

$$v^i = \tilde{v}^i \circ c.$$

By the definition of $c'(t)$,

$$c'(t)\tilde{v}^i = c_*\left(\frac{d}{dt}\right)\tilde{v}^i = \frac{d}{dt}(\tilde{v}^i \circ c) = \frac{d}{dt}v^i.$$

Therefore,

$$\begin{aligned} D_{c'(t)}\tilde{V} &= \sum (c'(t)\tilde{v}^i)\, \partial_i|_{c(t)} \qquad \text{(by (4.2))} \\ &= \sum \frac{dv^i}{dt}\, \partial_i|_{c(t)} = \frac{dV}{dt}. \end{aligned}$$

□

Problems

4.1. Lie bracket of vector fields
Let f, g be C^∞ functions and X, Y be C^∞ vector fields on a manifold M. Show that

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X.$$

(*Hint*: Two smooth vector fields V and W on a manifold M are equal if and only if for every $h \in C^\infty(M)$, $Vh = Wh$.)

4.2. Directional derivative in $\mathbb{R}^n$
Prove Proposition 4.3(i).

4.3. Directional derivative on a submanifold
Let M be a regular submanifold of $\mathbb{R}^n$ and

$$D\colon \mathfrak{X}(M) \times \Gamma(T\mathbb{R}^n|_M) \to \Gamma(T\mathbb{R}^n|_M)$$

the directional derivative on M. Since $\mathfrak{X}(M) \subset \Gamma(T\mathbb{R}^n|_M)$, we can restrict D to $\mathfrak{X}(M) \times \mathfrak{X}(M)$ to obtain

$$D\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(T\mathbb{R}^n|_M).$$

(a) Let T be the unit tangent vector field to the circle S^1. Prove that $D_T T$ is not tangent to S^1. This example shows that $D|_{\mathfrak{X}(M)\times\mathfrak{X}(M)}$ does not necessarily map into $\mathfrak{X}(M)$.
(b) If $X, Y \in \mathfrak{X}(M)$, prove that

$$D_X Y - D_Y X = [X,Y].$$

§5 The Shape Operator

To define the curvature of a plane curve in Section 2, we differentiated its unit tangent vector field with respect to arc length. Extrapolating from this, we can try to describe the curvature of a surface in $\mathbb{R}^3$ by differentiating a unit normal vector field along various directions.

5.1 Normal Vector Fields

Recall that a point p of $\mathbb{R}^n$ is a ***regular point*** of a function $f\colon \mathbb{R}^n \to \mathbb{R}$ if the differential $f_{*,p}\colon T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}$ is surjective, equivalently, if at least one partial derivative $\partial f/\partial x^i(p)$ is nonzero. A point $q \in \mathbb{R}$ is a ***regular value*** if its inverse image $f^{-1}(q)$ consists entirely of regular points; otherwise, it is a ***singular value***.

A ***hypersurface*** in $\mathbb{R}^n$ is the zero set $Z(f)$ of a C^∞ function f on $\mathbb{R}^n$. Consider a hypersurface M in $\mathbb{R}^3$ defined as the zero set of the C^∞ function $f(x,y,z)$. We assume that the partial derivatives f_x, f_y, f_z do not vanish simultaneously on M. Then 0 is a regular value of f, and $M = f^{-1}(0)$ is a regular level set. By the regular level set theorem, M is a regular submanifold of $\mathbb{R}^3$.

As a submanifold of $\mathbb{R}^3$, M inherits a Riemannian metric from the Euclidean metric on $\mathbb{R}^3$. Let $N = \operatorname{grad} f = \langle f_x, f_y, f_z\rangle$ be the gradient vector field of f on M.

Proposition 5.1. *If* 0 *is a regular value of the* C^∞ *function* f *on* $\mathbb{R}^3$*, then* $N = \operatorname{grad} f$ *is a nowhere-vanishing normal vector field along the smooth hypersurface* $M = Z(f)$*.*

Proof. Let $p \in M$ and $X_p \in T_pM$. It suffices to show that $\langle N_p, X_p\rangle = 0$. Choose a curve $c(t) = (x(t), y(t), z(t))$ in M with $c(0) = p$ and $c'(0) = X_p$ (such a curve always exists, for example by [21, Proposition 8.16]). Since $c(t)$ lies in M, $f(c(t)) = 0$ for all t. By the chain rule,

$$0 = \left.\frac{d}{dt}\right|_{t=0} f(c(t)) = f_x(p)x'(0) + f_y(p)y'(0) + f_z(p)z'(0).$$

Hence,

$$\langle N_p, X_p\rangle = \langle \operatorname{grad} f(p), c'(0)\rangle = 0. \qquad \square$$

By dividing $\operatorname{grad} f$ by its magnitude, we obtain a smooth unit normal vector field along the hypersurface M. Since a smooth surface in $\mathbb{R}^3$, orientable or not, is locally the zero set of a coordinate function, it follows from this proposition that a smooth unit normal vector field exists locally along any smooth surface in $\mathbb{R}^3$.

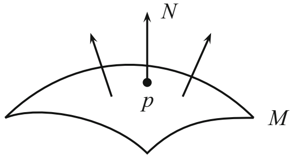


Fig. 5.1. A normal vector field near p.

5.2 The Shape Operator

Let p be a point on a surface M in $\mathbb{R}^3$ and let N be a C^∞ unit normal vector field on M (Figure 5.1). For any tangent vector $X_p \in T_pM$, define

$$L_p(X_p) = -D_{X_p}N.$$

This directional derivative makes sense, since N is defined along some curve with initial vector X_p. We put a negative sign in the definition of L_p so that other formulas such as Lemma 5.2 and Proposition 5.5 will be sign-free. Applying the vector X_p to $\langle N,N\rangle \equiv 1$ gives

$$\begin{aligned} 0 &= X_p\langle N,N\rangle \\ &= \langle D_{X_p}N, N_p\rangle + \langle N_p, D_{X_p}N\rangle \quad \text{(compatibility with the metric, Proposition 4.3)} \\ &= 2\langle D_{X_p}N, N_p\rangle. \end{aligned}$$

Thus, $D_{X_p}N$ is perpendicular to N_p at p and is therefore in the tangent plane T_pM. So L_p is a linear map $T_pM \to T_pM$. It is called the ***shape operator*** or the ***Weingarten map*** of the surface M at p. Note that the shape operator depends on the unit normal vector field N and the point p. With the unit normal vector field N on M fixed, as the point p varies in M, there is a different shape operator L_p at each p. To avoid cumbersome notation, we will usually omit the subscript p.

The shape operator, being the directional derivative of a unit normal vector field on a surface, should encode in it information about how the surface bends at p.

Lemma 5.2. *Let M be a surface in $\mathbb{R}^3$ having a C^∞ unit normal vector field N. For $X,Y \in \mathfrak{X}(M)$,*

$$\langle L(X),Y\rangle = \langle D_XY,N\rangle.$$

Proof. Since Y is tangent to M, the inner product $\langle Y,N\rangle$ is identically zero on U. Differentiating the equation $\langle Y,N\rangle \equiv 0$ with respect to X yields

$$\begin{aligned} 0 = X\langle Y,N\rangle &= \langle D_XY,N\rangle + \langle Y,D_XN\rangle \\ &= \langle D_XY,N\rangle - \langle Y,L(X)\rangle. \end{aligned}$$

Hence,

$$\langle L(X),Y\rangle = \langle D_XY,N\rangle. \qquad \square$$

In general, if $e_1, \dots, e_n$ is a basis for a vector space V and

$$v = \sum_i v^i e_i \in V, \text{ with } v^i \in \mathbb{R}, \tag{5.1}$$

then v^i is called the ***component*** of v in the e_i-direction. In case $e_1, \dots, e_n$ is an orthonormal basis in an inner product space $(V, \langle\ ,\ \rangle)$, by taking the inner product of both sides of (5.1) with e_j, one obtains

$$\langle v, e_j \rangle = \sum_i v^i \langle e_i, e_j \rangle = \sum_i v^i \delta_{ij} = v^j.$$

Thus, the component v^j of v in the e_j-direction is simply the inner product of v with e_j. In view of this, the right-hand side of Lemma 5.2 is the normal component of the directional derivative $D_{X_p} Y$.

Note that any tangent vector $X_p \in T_pM$ can be extended to a C^∞ vector field in a neighborhood of p: if $(U, x^1, \dots, x^n)$ is a chart containing p and $X_p = \sum a^i\, \partial/\partial x^i|_p$ for some $a^i \in \mathbb{R}$, then $X = \sum a^i\, \partial/\partial x^i$ is such a vector field on U.

Proposition 5.3. *The shape operator is self-adjoint: for any $X_p, Y_p \in T_pM$,*

$$\langle L(X_p), Y_p \rangle = \langle X_p, L(Y_p) \rangle.$$

Proof. Suppose the C^∞ unit normal vector field N is defined on a neighborhood U of p. Let X, Y be vector fields on U that extend the vectors X_p, Y_p at p. By Lemma 5.2,

$$\langle L(X), Y \rangle = \langle D_X Y, N \rangle. \tag{5.2}$$

Similarly, reversing the roles of X and Y, we have

$$\langle L(Y), X \rangle = \langle D_Y X, N \rangle. \tag{5.3}$$

By Problem 4.3(b),

$$D_X Y - D_Y X = [X, Y]. \tag{5.4}$$

Combining the three preceding equations (5.2), (5.3), and (5.4), we get

$$\begin{aligned} &\langle L(X), Y \rangle - \langle L(Y), X \rangle \\ &= \langle D_X Y, N \rangle - \langle D_Y X, N \rangle \\ &= \langle D_X Y - D_Y X, N \rangle \\ &= \langle [X, Y], N \rangle && \text{(by (5.4))} \\ &= 0 && \text{(since } [X, Y] \text{ is a tangent vector field on } M). \end{aligned}$$

Hence,

$$\langle L(X), Y \rangle = \langle L(Y), X \rangle = \langle X, L(Y) \rangle.$$

□

If $T\colon V \to W$ is a linear map between two vector spaces V and W, $\mathcal{B}_V = \{v_1,\dots,v_n\}$ a basis for V, and $\mathcal{B}_W = \{w_1,\dots,w_m\}$ a basis for W, then

$$T(v_j) = \sum_{i=1}^{m} a^i_j w_i, \quad j = 1,\dots,n,$$

for a unique matrix $[a^i_j]$, called the ***matrix*** of the linear map T with respect to the bases $\mathcal{B}_V$ and $\mathcal{B}_W$.

It follows from Proposition 5.3 that the matrix of the shape operator with respect to an orthonormal basis for T_pM is symmetric, for if e_1, e_2 is an orthonormal basis for T_pM and

$$\begin{aligned} L(e_1) &= ae_1 + be_2, \\ L(e_2) &= he_1 + ce_2, \end{aligned}$$

then

$$b = \langle L(e_1), e_2\rangle = \langle e_1, L(e_2)\rangle = h.$$

Since the eigenvalues of a symmetric matrix are real (Problem 5.1), the shape operator has real eigenvalues. We will see shortly the meaning of these eigenvalues.

5.3 Curvature and the Shape Operator

Consider as before a surface M in $\mathbb{R}^3$ having a C^∞ unit normal vector field N. Lemma 5.2 on the shape operator has a counterpart for vector fields along a curve.

Proposition 5.4. *Let $c\colon [a,b] \to M$ be a curve in M and let V be a vector field in M along c. Then*

$$\langle L(c'(t)), V\rangle = \left\langle \frac{dV}{dt}, N_{c(t)} \right\rangle.$$

Remark. When we write V or dV/dt, we mean $V(t)$ and $dV(t)/dt$, respectively.

Proof. Since $V(t)$ is tangent to M, the inner product $\langle V(t), N_{c(t)}\rangle$ is identically zero. By Proposition 4.7, differentiating with respect to t yields

$$\begin{aligned} 0 &= \frac{d}{dt}\langle V(t), N_{c(t)}\rangle \\ &= \left\langle \frac{dV}{dt}, N_{c(t)} \right\rangle + \left\langle V(t), \frac{d}{dt}N_{c(t)} \right\rangle \\ &= \left\langle \frac{dV}{dt}, N_{c(t)} \right\rangle + \langle V(t), D_{c'(t)}N\rangle \qquad \text{(by Proposition 4.11).} \end{aligned}$$

Thus,

$$\langle V(t), L(c'(t))\rangle = \langle V(t), -D_{c'(t)}N\rangle = \left\langle \frac{dV}{dt}, N_{c(t)} \right\rangle. \qquad \square$$

Proposition 5.5. *Suppose $\gamma(s)$ is a normal section, parametrized by arc length, determined by a unit tangent vector $X_p \in T_pM$ and the unit normal vector N_p. Then the normal curvature of $\gamma(s)$ with respect to N_p at p is given by the second fundamental form:*

$$\kappa(X_p) = \langle L(X_p), X_p \rangle = \mathrm{II}(X_p, X_p).$$

Proof. By definition, $\gamma(0) = p$ and $\gamma'(0) = X_p$. Let $T(s) := \gamma'(s)$ be the unit tangent vector field along $\gamma(s)$. Then the curvature of the normal section $\gamma(s)$ is

$$\begin{aligned}
\kappa\big(\gamma'(s)\big) &= \langle \gamma''(s), N_{\gamma(s)} \rangle \\
&= \langle dT/ds, N_{\gamma(s)} \rangle \\
&= \langle L\big(\gamma'(s)\big), T \rangle \qquad \text{(by Proposition 5.4)} \\
&= \langle L(T), T \rangle.
\end{aligned}$$

Evaluating at $s = 0$ gives

$$\kappa(X_p) = \langle L(X_p), X_p \rangle = \mathrm{II}(X_p, X_p). \qquad \square$$

Proposition 5.6. *The principal directions of the surface M in $\mathbb{R}^3$ at p are the unit eigenvectors of the shape operator L; the principal curvatures are the eigenvalues of L.*

Proof. The principal curvatures at p are the maximum and minimum values of the function

$$\kappa(X_p) = \mathrm{II}(X_p, X_p) = \langle L(X_p), X_p \rangle$$

for $X_p \in T_pM$ satisfying $\langle X_p, X_p \rangle = 1$. This type of optimization problem with a constraint lends itself to the method of the Lagrange multiplier from vector calculus.

Choose an orthonormal basis e_1, e_2 for T_pM so that

$$X_p = xe_1 + ye_2 = \begin{bmatrix} x \\ y \end{bmatrix}.$$

By Proposition 5.3 the matrix of L relative to the basis e_1, e_2 is a symmetric matrix

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

meaning

$$\begin{aligned}
L(e_1) &= ae_1 + be_2, \\
L(e_2) &= be_1 + ce_2.
\end{aligned}$$

In matrix notation,

$$\begin{aligned}
L(X_p) = L(xe_1 + ye_2) &= (ax + by)e_1 + (bx + cy)e_2 \\
&= \begin{bmatrix} ax + by \\ bx + cy \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = AX_p,
\end{aligned}$$

and

$$\kappa(X_p) = \langle L(X_p), X_p \rangle = \langle AX_p, X_p \rangle = X_p^T AX_p$$
$$= ax^2 + 2bxy + cy^2.$$

The problem of finding the principal curvatures becomes a standard calculus problem: find the maximum and minimum of the function

$$\kappa(X_p) = ax^2 + 2bxy + cy^2 = X_p^T AX_p$$

subject to the constraint

$$g(X_p) = \langle X_p, X_p \rangle = x^2 + y^2 = 1.$$

Now

$$\operatorname{grad} \kappa = \begin{bmatrix} 2ax + 2by \\ 2bx + 2cy \end{bmatrix} = 2AX_p$$

and

$$\operatorname{grad} g = 2X_p.$$

By the method of the Lagrange multiplier, at the maximum or minimum of κ, there is a scalar λ such that

$$\operatorname{grad} \kappa = \lambda \operatorname{grad} g, \quad \text{or } AX_p = \lambda X_p.$$

Thus, the maximum and minimum of κ occur at unit eigenvectors of A. These are the principal directions at p.

Let X_p be a principal direction at p. Then the corresponding principal curvature is the normal curvature

$$\kappa(X_p) = \langle L(X_p), X_p \rangle = \langle AX_p, X_p \rangle = \langle \lambda X_p, X_p \rangle = \lambda \langle X_p, X_p \rangle = \lambda,$$

the eigenvalue associated to the eigenvector X_p. □

Corollary 5.7. (i) *The Gaussian curvature of a surface M in $\mathbb{R}^3$ is the determinant of the shape operator.*

(ii) *If e_1, e_2 is an orthonormal basis for the tangent space T_pM of the surface M, then the Gaussian curvature at p is*

$$K = \langle L(e_1), e_1 \rangle \langle L(e_2), e_2 \rangle - \langle L(e_1), e_2 \rangle \langle L(e_2), e_1 \rangle.$$

Proof. (i) The determinant of a linear map is the product of its eigenvalues (Problem 5.2). For the shape operator L the eigenvalues are the principal curvatures κ_1, κ_2. So $\det L = \kappa_1 \kappa_2 = K$, the Gaussian curvature.

(ii) If

$$L(e_1) = ae_1 + be_2 \quad \text{and} \quad L(e_2) = be_1 + ce_2,$$

then

$$a = \langle L(e_1), e_1 \rangle, \;\; b = \langle L(e_1), e_2 \rangle = \langle L(e_2), e_1 \rangle, \;\; c = \langle L(e_2), e_2 \rangle.$$

The matrix of L relative to the orthonormal basis e_1, e_2 is

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

Hence,

$$\begin{aligned} \det L &= ac - b^2 \\ &= \langle L(e_1), e_1\rangle\langle L(e_2), e_2\rangle - \langle L(e_1), e_2\rangle\langle L(e_2), e_1\rangle. \end{aligned}$$ □

5.4 The First and Second Fundamental Forms

A point p of a smooth surface M in $\mathbb{R}^3$, the Euclidean Riemannian metric is a symmetric bilinear form on the tangent space T_pM. It is called the ***first fundamental form*** of M at p. If $L\colon T_pM \to T_pM$ is the shape operator, the symmetric bilinear form on T_pM

$$\mathrm{II}(X_p, Y_p) = \langle L(X_p), Y_p\rangle$$

is called the ***second fundamental form*** of the surface M at p. The first fundamental form is the metric and the second fundamental form is essentially the shape operator. These two fundamental forms encode in them much of the geometry of the surface M.

Let e_1, e_2 be a basis for the tangent space T_pM. We set

$$E := \langle e_1, e_1\rangle, \quad F := \langle e_1, e_2\rangle, \quad G := \langle e_2, e_2\rangle.$$

If $X_p = x^1 e_1 + x^2 e_2$ and $Y_p = y^1 e_1 + y^2 e_2$, then

$$\begin{aligned} \langle X_p, Y_p\rangle &= \langle e_1, e_1\rangle x^1 y^1 + \langle e_1, e_2\rangle x^1 y^2 \\ &\qquad + \langle e_2, e_1\rangle x^2 y^1 + \langle e_2, e_2\rangle x^2 y^2 \\ &= Ex^1y^1 + Fx^1y^2 + Fx^2y^1 + Gx^2y^2 \\ &= [x^1\ x^2] \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix}. \end{aligned}$$

Thus, the three numbers E, F, G determine completely the first fundamental form of M at p. They are called the ***coefficients of the first fundamental form*** relative to e_1, e_2.

Similarly, the three numbers

$$e := \mathrm{II}(e_1, e_1), \quad f := \mathrm{II}(e_1, e_2), \quad g := \mathrm{II}(e_2, e_2)$$

determine completely the second fundamental form of M at p:

$$\mathrm{II}(X_p, Y_p) = [x^1\ x^2] \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix}.$$

They are called the ***coefficients of the second fundamental form*** relative to e_1, e_2. As p varies in an open set U, if e_1, e_2 remain a basis of T_pM at each point, these coefficients are six functions on U. Classically, M is taken to be a coordinate patch (U, u, v) with $e_1 = \partial/\partial u$ and $e_2 = \partial/\partial v$, and the differential geometry of M is done in terms of the six functions E, F, G, e, f, g.

Theorem 5.8. *Suppose M and M' are two smooth Riemannian manifolds of dimension 2, and $\varphi\colon M \to M'$ is a diffeomorphism. Let E, F, G be the coefficients of the first fundamental form relative to a frame e_1, e_2 on M, and E', F', G' the corresponding coefficients relative to $e_1' := \varphi_* e_1$, $e_2' := \varphi_* e_2$ on M'. Then φ is an isometry if and only if E, F, G at p are equal to E', F', G' at $\varphi(p)$, respectively, for all $p \in M$.*

Proof. The diffeomorphism φ is an isometry if and only if

$$\langle \varphi_* X_p, \varphi_* Y_p \rangle_{\varphi(p)} = \langle X_p, Y_p \rangle_p.$$

This condition holds if and only if

$$\langle \varphi_* e_i, \varphi_* e_j \rangle_{\varphi(p)} = \langle e_i, e_j \rangle_p.$$

for all $i = 1, 2$ and $j = 1, 2$, i.e.,

$$E'\big(\varphi(p)\big) = E(p), \quad F'\big(\varphi(p)\big) = F(p), \quad G'\big(\varphi(p)\big) = G(p). \qquad \square$$

5.5 The Catenoid and the Helicoid

The graph of the hyperbolic cosine function

$$y = \cosh x := \frac{e^x + e^{-x}}{2}$$

is called a ***catenary*** (Figure 5.2).

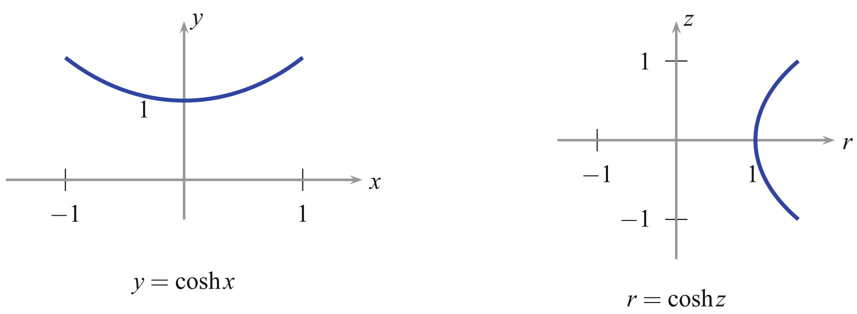


Fig. 5.2. Catenaries



Fig. 5.3. A hanging chain

Using physics, it can be shown that a hanging chain naturally assumes the shape of a catenary (Figure 5.3).

The surface of revolution obtained by rotating the catenary $r = \cosh z$ about the z-axis is called a ***catenoid*** (Figure 5.4). It has parametrization

$$(r\cos\theta, r\sin\theta, z) = \big((\cosh u)\cos\theta, (\cosh u)\sin\theta, u\big),$$

where we set $z = u$.

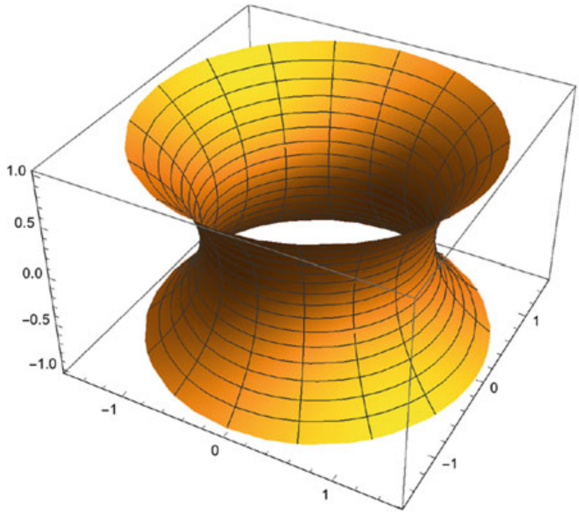


Fig. 5.4. A catenoid

The ***helicoid*** is the surface with parametrization

$$(r\cos\theta, r\sin\theta, \theta), \quad -\infty < r, \theta < \infty;$$

it is the surface traced out by a horizontal stick moving upward with constant speed while rotating about a vertical axis through its midpoint (Figure 5.5).

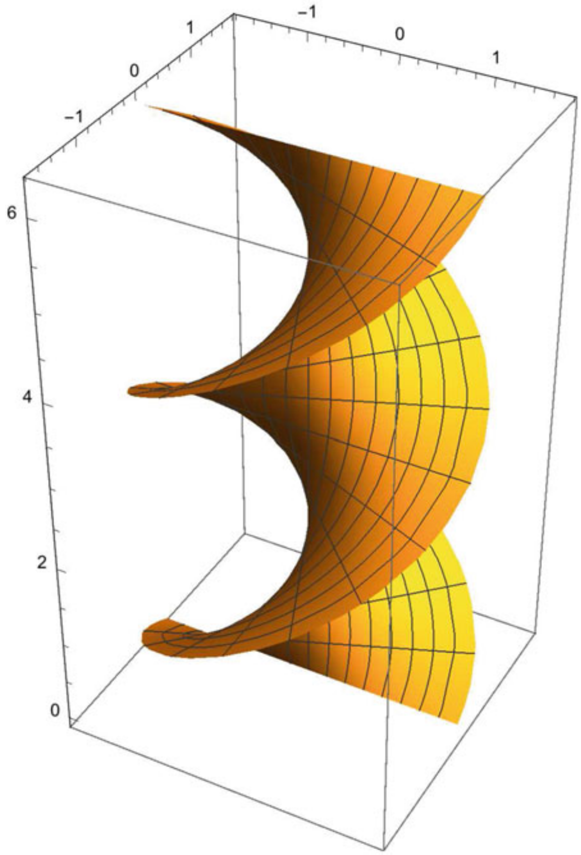


Fig. 5.5. A helicoid

The catenoid shown in Figure 5.4 has

$$-1 \le u \le 1 \quad 0 \le \theta \le 2\pi.$$

If we remove from this catenoid the points with $u = -1, 1$ and $\theta = 0, 2\pi$, then what is left is a coordinate chart U.

Similarly, the helicoid in Figure 5.5 has

$$-a \le r \le a, \quad 0 \le \theta \le 2\pi$$

for some positive real number a. We remove from it points with $r = 0, a$ and $\theta = 0, 2\pi$ to obtain a coordinate chart U'.

As coordinate charts, there are diffeomorphisms $U \simeq (-1,1) \times (0,2\pi)$ and $U' \simeq (-a,a) \times (0,2\pi)$. Since $(-1,1)$ is diffeomorphic to $(-\sinh 1, \sinh 1)$ via the map $u \mapsto \sinh h$, there is a diffeomorphism $\varphi\colon U \to U'$ given by

$$\big((\cosh u)\cos\theta, (\cosh u)\sin\theta, u\big) \mapsto \big((\sinh u)\cos\theta, (\sinh u)\sin\theta, \theta\big).$$

By computing the three functions E, F, G relative to the frame $e_1 = \partial/\partial u$, $e_2 = \partial/\partial\theta$ on the catenoid and the frame $e_1' = \varphi_* e_1$, $e_2' = \varphi_* e_2$ on the helicoid, respectively, one can show that $\varphi\colon U \to U'$ is an isometry (Problem 5.11). Physically, what this means is that if one cuts the catenoid in Figure 5.4 along a meridian, straighten out the meridian, and let the catenoid hang, it will assume the shape of the helicoid in Figure 5.5.

Problems

5.1. Eigenvalues and eigenvectors of a symmetric matrix
Suppose A is an $n \times n$ real symmetric matrix with complex eigenvector X and corresponding complex eigenvalue λ. Think of X as a column vector and consider the complex number $\bar{X}^{\mathrm{T}} AX$, where $\bar{X}$ is the complex conjugate of X and the superscript T means the transpose.

(a) Show that $\bar{X}^{\mathrm{T}} A = \bar{\lambda}\bar{X}^{\mathrm{T}}$.
(b) By computing the complex number $\bar{X}^{\mathrm{T}} AX$ in two ways, as $(\bar{X}^{\mathrm{T}} A)X$ and as $\bar{X}^{\mathrm{T}}(AX)$, show that $\lambda = \bar{\lambda}$. This proves that the eigenvalues of a real symmetric matrix are always real.
(c) Show that A has n independent real eigenvectors.
(d) Show that eigenvectors of A corresponding to distinct eigenvalues are orthogonal. (*Hint*: Let X and Y be eigenvectors corresponding to distinct eigenvalues λ and μ, respectively. Note that $\langle X, Y\rangle = X^{\mathrm{T}} Y$. Compute $X^{\mathrm{T}} AY$ two different ways.)

Part (d) shows that if the principal curvatures at a point of a surface in $\mathbb{R}^3$ are distinct, then the principal directions are orthogonal.

5.2. Determinant and eigenvalues
Prove that the determinant of any $n \times n$ matrix, real or complex, is the product of its eigenvalues. (*Hint*: Triangularize the matrix.)

5.3. The Gauss map
Let M be a smooth oriented surface in $\mathbb{R}^3$, with a smooth unit normal vector field N. The ***Gauss map*** of M is the map

$$\nu\colon M \to S^2 \subset \mathbb{R}^3, \qquad \nu(p) = N_p,$$

where N_p is considered to be a unit vector starting at the origin. Show that the differential of the Gauss map at p is the negative of the Weingarten map:

$$\nu_{*,p}(X_p) = -L(X_p)$$

for any $X_p \in T_pM$. (*Hint*: Compute the differential by using curves. See [21, Section 8.7].)

5.4. Total curvature

The ***total curvature*** of a smooth oriented surface M in $\mathbb{R}^3$ is defined to be the integral $\int_M K$, if it exists, of the Gaussian curvature K. Prove that the total curvature of M is, up to sign, the area of the image of the Gauss map:

$$\int_M K = \text{Area of } \nu(M).$$

5.5. Shape operator of a cylinder

Let M be the cylinder of radius a in $\mathbb{R}^3$ defined by $x^2 + y^2 = a^2$. At $p = (x,y,z) \in M$, the vectors

$$e_1 = \frac{1}{a}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right), \quad e_2 = \frac{\partial}{\partial z}$$

form an orthonormal basis for the tangent space T_pM. Let N be the unit normal vector field

$$N = \frac{1}{a}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right).$$

(a) Find the matrix of the shape operator with respect to the basis e_1, e_2 at p. (Compute $L(e_j) = -D_{e_j}N$ and find the matrix $[a^i_j]$ such that $L(e_j) = \sum a^i_j e_i$.)
(b) Compute the mean and Gaussian curvatures of the cylinder M.

5.6. Shape operator of a sphere

Let M be the sphere of radius a in $\mathbb{R}^3$ defined by

$$x^2 + y^2 + z^2 = a^2.$$

Parametrize the sphere using spherical coordinates:

$$\begin{aligned} x &= a\sin\phi\cos\theta \\ y &= a\sin\phi\sin\theta \\ z &= a\cos\phi, \qquad 0 \le \phi < \pi, \quad 0 \le \theta < 2\pi. \end{aligned}$$

Then for each $p \in M$, $e_1 = \partial/\partial\phi$, $e_2 = \partial/\partial\theta$ is a basis for the tangent space T_pM for $p \in M$. Let N_p be the unit outward normal vector at p on the sphere.

(a) Find the matrix of the shape operator of the sphere with respect to the basis e_1, e_2 at p.
(b) Compute the mean and Gaussian curvatures of the sphere M at p using (a).

5.7. Surface of revolution in $\mathbb{R}^3$

Let $\big(f(u), g(u)\big)$ be a unit-speed curve without self-intersection in the (y,z)-plane. Assume $f(u) > 0$, so that $\big(f(u), g(u)\big)$ can be rotated about the z-axis to form a surface of revolution M in $\mathbb{R}^3$. A parametrization of the surface of revolution is

$$\psi(u,v) = \begin{bmatrix} f(u)\cos v \\ f(u)\sin v \\ g(u) \end{bmatrix}, \quad 0 < v < 2\pi.$$

Assume that ψ is a diffeomorphism onto its image, so that u, v are coordinates on a chart U in M. Then $e_1 = \partial/\partial u$, $e_2 = \partial/\partial v$ is a basis for the tangent space $T_p(M)$ for $p \in U$.

(a) Find the matrix of the shape operator of the surface of revolution with respect to the basis e_1, e_2 at p.
(b) Compute the mean and Gaussian curvature of the surface of revolution M at p.

5.8. The average value of normal curvature
Let M be an oriented surface in $\mathbb{R}^3$ with a unit normal vector field N, and let p be a point in M. Each unit tangent vector $X_p \in T_pM$ determines a normal section with curvature $\kappa(X_p)$ at p. The unit tangent vectors in T_pM form a circle S^1 and the normal curvature κ is a function on this circle. Show that the average value $(1/2\pi)\int_{S^1} \kappa$ of the normal curvature at p is the mean curvature at p. (In general, the ***average value*** of a function f on the unit circle S^1 is $(1/2\pi)\int_{S^1} f$.)

5.9. The coefficients of the two fundamental forms
Suppose M is an oriented surface in $\mathbb{R}^3$, $p \in M$, and e_1, e_2 is a basis for T_pM. Let E, F, G, e, f, g be the coefficients of the first and second fundamental forms of M at p relative to e_1, e_2.

(a) Show that the matrix of the shape operator $L\colon T_pM \to T_pM$ relative to e_1, e_2 is

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} e & f \\ f & g \end{bmatrix}.$$

(b) Compute the Gaussian curvature K and the mean curvature H of M at p in terms of E, F, G, e, f, g.

5.10. The first and second fundamental forms of a graph
Let M be the graph of a C^∞ function $z = h(x,y)$ in $\mathbb{R}^3$. Then M can be parametrized by $\sigma(x,y) = \big(x, y, h(x,y)\big)$. Let $e_1 = \sigma_*(\partial/\partial x)$ and $e_2 = \sigma_*(\partial/\partial y)$.

(a) Show that the coefficients of the first fundamental form relative to e_1, e_2 are

$$E = 1 + h_x^2, \quad F = h_x h_y, \quad G = 1 + h_y^2.$$

(Here $h_x = \partial h/\partial x$.)
(b) Compute the coefficients of the second fundamental forms relative to e_1, e_2.
(c) Show that the Gaussian curvature of the graph $z = h(x,y)$ is

$$K = \frac{h_{xx}h_{yy} - h_{xy}^2}{(h_x^2 + h_y^2 + 1)^2}.$$

5.11. Isometry between a catenoid and a helicoid
Let U be the open set in the catenoid with parametrization

$$(x,y,z) = \big((\cosh u)\cos\theta, (\cosh u)\sin\theta, u\big), \quad -1 < u < 1, \quad 0 < \theta < 2\pi,$$

and let U' be the open set in the helicoid with parametrization

$$(x,y,z) = (u'\cos\theta, u'\sin\theta, \theta), \quad -\sinh 1 < u' < \sinh 1, \quad 0 < \theta < 2\pi.$$

Define $\varphi\colon U \to U'$ by

$$\varphi\big((\cosh u)\cos\theta, (\cosh u)\sin\theta, u\big) = \big((\sinh u)\cos\theta, (\sinh u)\sin\theta, \theta\big).$$

As explained in Section 5.5, φ is a diffeomorphism.

Viewing u, θ as coordinate functions on U, we let $e_1 = \partial/\partial u$, $e_2 = \partial/\partial\theta$, which are then tangent vector fields on U.

(a) Compute e_1, e_2 in terms of $\partial/\partial x, \partial/\partial y, \partial/\partial z$.
(b) Compute E, F, G relative to the frame e_1, e_2 at $p \in U$ of the catenoid.
(c) Let $e_1' = \varphi_* e_1, e_2' = \varphi_* e_2$ on the open set U' of the helicoid. Compute e_1', e_2' in terms of $\partial/\partial x, \partial/\partial y, \partial/\partial z$ at $\varphi(p) \in U'$.
(d) Compute E', F', G' relative to the frame e_1', e_2'.
(e) Prove that $\varphi\colon U \to U'$ is an isometry.

§6 Affine Connections

Consider a smooth vector field Y on a manifold M and a tangent vector $X_p \in T_pM$ at a point p in M. To define the directional derivative of Y in the direction X_p, it is necessary to compare the values of Y in a neighborhood of p. If q is a point near p, in general it is not possible to compare the vectors Y_q and Y_p by taking the difference $Y_q - Y_p$, since they are in distinct tangent spaces. For this reason, the directional derivative of a vector field on an arbitrary manifold M cannot be defined in the same way as in Section 4. Instead, we extract from the directional derivative in $\mathbb{R}^n$ certain key properties and call any map $D\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ with these properties an ***affine connection***. Intuitively, an affine connection on a manifold is simply a way of differentiating vector fields on the manifold.

Mimicking the directional derivative in $\mathbb{R}^n$, we define the torsion and curvature of an affine connection on a manifold M. Miraculously, both torsion and curvature are linear over C^∞ functions in every argument.

We will see in a later section that there are infinitely many affine connections on any manifold. On a Riemannian manifold, however, there is a unique torsion-free affine connection compatible with the metric, called the ***Riemannian*** or ***Levi-Civita connection***. As an example, we describe the Riemannian connection on a surface in $\mathbb{R}^3$.

6.1 Affine Connections

On an arbitrary manifold M, which is not necessarily embedded in a Euclidean space, we can define the directional derivative of a C^∞ function f in the direction $X_p \in T_pM$ in the same way as before:

$$\nabla_{X_p} f = X_p f.$$

However, there is no longer a canonical way to define the directional derivative of a vector field Y. Formula (4.2) fails because unlike in $\mathbb{R}^n$, there is no canonical basis for the tangent space T_pM.

Whatever definition of directional derivative of a vector field one adopts, it should satisfy the properties in Proposition 4.2. This motivates the following definition.

Definition 6.1. An ***affine connection*** on a manifold M is an $\mathbb{R}$-bilinear map

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),$$

written $\nabla_X Y$ for $\nabla(X,Y)$, satisfying the two properties below: if $\mathcal{F}$ is the ring $C^\infty(M)$ of C^∞ functions on M, then for all $X, Y \in \mathfrak{X}(M)$,

(i) $\nabla_X Y$ is $\mathcal{F}$-linear in X,
(ii) (Leibniz rule) $\nabla_X Y$ satisfies the Leibniz rule in Y: for $f \in \mathcal{F}$,

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y.$$

Example 6.2. The directional derivative D_X of a vector field Y on $\mathbb{R}^n$ is an affine connection on $\mathbb{R}^n$, sometimes called the ***Euclidean connection*** on $\mathbb{R}^n$.

6.2 Torsion and Curvature

Given an affine connection ∇ on a manifold M, one might ask whether it satisfies the same properties as Proposition 4.3 for the Euclidean connection on $\mathbb{R}^n$. For $X, Y \in \mathfrak{X}(M)$, set

$$\begin{aligned} T(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y] \in \mathfrak{X}(M), \\ R(X,Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \\ &= \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \operatorname{End}(\mathfrak{X}(M)). \end{aligned}$$

We call T the ***torsion*** and R the ***curvature*** of the connection. There does not seem to be a good reason for calling $T(X,Y)$ the torsion, but as we shall see in Section 8, $R(X,Y)$ is intimately related to the Gaussian curvature of a surface.

An affine connection ∇ on a manifold M gives rise to a linear map

$$\mathfrak{X}(M) \to \operatorname{End}_{\mathbb{R}}(\mathfrak{X}(M)), \quad X \mapsto \nabla_X.$$

Here both vector spaces $\mathfrak{X}(M)$ and $\operatorname{End}_{\mathbb{R}}(\mathfrak{X}(M))$ are Lie algebras. The curvature measures the deviation of the map $X \mapsto \nabla_X$ from being a Lie algebra homomorphism.

Recall that $\mathcal{F}$ is the ring $C^\infty(M)$ of C^∞ functions on the manifold M. Although $\nabla_X Y$ is not $\mathcal{F}$-linear in Y, it turns out, amazingly, that both torsion and curvature are $\mathcal{F}$-linear in all their arguments.

Proposition 6.3. *Let X, Y, Z be smooth vector fields on a manifold M with affine connection ∇.*

(i) *The torsion $T(X,Y)$ is $\mathcal{F}$-linear in X and in Y.*
(ii) *The curvature $R(X,Y)Z$ is $\mathcal{F}$-linear in X, Y, and Z.*

Proof. We will first check the $\mathcal{F}$-linearity of $R(X,Y)Z$ in X. For this, it is useful to recall the following formula from the theory of manifolds (Problem 4.1): if $f, g \in \mathcal{F} = C^\infty(M)$, then

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X. \tag{6.1}$$

By the definition of curvature,

$$R(fX,Y)Z = \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z. \tag{6.2}$$

By the $\mathcal{F}$-linearity of ∇ in X, the first term in (6.2) is $f\nabla_X\nabla_Y Z$, and by the Leibniz rule, the second term is

$$\nabla_Y(f\nabla_X Z) = (Yf)\nabla_X Z + f\nabla_Y\nabla_X Z.$$

Applying (6.1), the last term in (6.2) is

$$\begin{aligned} \nabla_{[fX,Y]} Z &= \nabla_{f[X,Y]-(Yf)X} Z \\ &= f\nabla_{[X,Y]} Z - (Yf)\nabla_X Z, \end{aligned}$$

since $X(1) = 0$. Combining the three terms gives

$$\begin{aligned} R(fX,Y)Z &= f(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z) \\ &= fR(X,Y)Z. \end{aligned}$$

Because $R(X,Y)$ is skew-symmetric in X and Y, $\mathcal{F}$-linearity in Y follows from $\mathcal{F}$-linearity in the first argument:

$$R(X, fY) = -R(fY,X) = -fR(Y,X) = fR(X,Y).$$

We leave the $\mathcal{F}$-linearity of $T(X,Y)$ as well as that of $R(X,Y)Z$ in Z as exercises (Problems 6.2 and 6.3). □

6.3 The Riemannian Connection

To narrow down the number of affine connections on a manifold, we impose additional conditions on a connection. On any manifold, we say that a connection is ***torsion-free*** if its torsion is zero. On a Riemannian manifold, we say that a connection is ***compatible with the metric*** if for all $X,Y,Z \in \mathfrak{X}(M)$,

$$Z\langle X,Y\rangle = \langle \nabla_Z X,Y\rangle + \langle X,\nabla_Z Y\rangle.$$

It turns out that these two additional conditions are enough to determine a connection uniquely on a Riemannian manifold.



Tullio Levi-Civita (1873–1941)

Definition 6.4. On a Riemannian manifold a ***Riemannian connection***, sometimes called a ***Levi-Civita connection***, is an affine connection that is torsion-free and compatible with the metric.

Lemma 6.5. *A C^∞ vector field X on a Riemannian manifold $(M,\langle\ ,\ \rangle)$ is uniquely determined by the values $\langle X,Z\rangle$ for all $Z \in \mathfrak{X}(M)$.*

Proof. We need to show that if $X' \in \mathfrak{X}(M)$ and $\langle X,Z\rangle = \langle X',Z\rangle$ for all $Z \in \mathfrak{X}(M)$, then $X = X'$. With $Y = X - X'$, this is equivalent to showing that if $\langle Y,Z\rangle = 0$ for all $Z \in \mathfrak{X}(M)$, then $Y = 0$. Take $Z = Y$. By the positive-definiteness of the inner product at each point $p \in M$,

$$\begin{aligned} \langle Y,Y\rangle = 0 \ &\Rightarrow\ \langle Y_p,Y_p\rangle = 0 \text{ for all } p \in M \\ &\Rightarrow\ Y_p = 0 \text{ for all } p \in M \\ &\Rightarrow\ Y = 0. \end{aligned}$$

□

Theorem 6.6. *On a Riemannian manifold there is a unique Riemannian connection.*

Proof. Suppose ∇ is a Riemannian connection on M. By Lemma 6.5, to specify $\nabla_X Y$, it suffices to know $\langle \nabla_X Y, Z\rangle$ for every vector field $Z \in \mathfrak{X}(M)$. So we will try to find a formula for $\langle \nabla_X Y, Z\rangle$ involving only the Riemannian metric and canonical operations on vector fields such as the Lie bracket.

A Riemannian connection satisfies the two formulas

$$\nabla_X Y - \nabla_Y X - [X,Y] = 0 \tag{6.3}$$

and

$$X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle. \tag{6.4}$$

Cyclically permuting X, Y, Z in (6.4) gives two other formulas:

$$Y\langle Z,X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle, \tag{6.5}$$

$$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle. \tag{6.6}$$

Using (6.3) we can rewrite $\nabla_Y X$ in (6.5) in terms of $\nabla_X Y$:

$$Y\langle Z,X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_X Y\rangle - \langle Z, [X,Y]\rangle. \tag{6.7}$$

Subtracting (6.6) from (6.4) and then adding (6.7) to it will create terms involving $\nabla_X Z - \nabla_Z X$ and $\nabla_Y Z - \nabla_Z Y$, which are equal to $[X,Z]$ and $[Y,Z]$ by torsion-freeness:

$$\begin{aligned} X\langle Y,Z\rangle &+ Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ &= 2\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z - \nabla_Z X\rangle + \langle X, \nabla_Y Z - \nabla_Z Y\rangle - \langle Z, [X,Y]\rangle \\ &= 2\langle \nabla_X Y, Z\rangle + \langle Y, [X,Z]\rangle + \langle X, [Y,Z]\rangle - \langle Z, [X,Y]\rangle. \end{aligned}$$

Solving for $\langle \nabla_X Y, Z\rangle$, we get

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle = {} & X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ & - \langle X, [Y,Z]\rangle + \langle Y, [Z,X]\rangle + \langle Z, [X,Y]\rangle. \end{aligned} \tag{6.8}$$

This formula proves that a Riemannian connection, if it exists, is unique.

Define $\nabla_X Y$ by the formula (6.8). It is a straightforward exercise to check that so defined, ∇ is a torsion-free affine connection compatible with the metric (Problem 6.4). This proves the existence of a Riemannian connection on a Riemannian manifold. □

Example 6.7. By Proposition 4.3, the Riemannian connection on $\mathbb{R}^n$ with its usual Euclidean metric is the directional derivative $D_X Y$ on $\mathbb{R}^n$.

6.4 Orthogonal Projection on a Surface in $\mathbb{R}^3$

Suppose M is a smooth surface in $\mathbb{R}^3$, which we do not assume to be orientable. At each point $p \in M$, let ν_p be the normal line through p. It is perpendicular to the tangent space T_pM, so there is an orthogonal decomposition

$$T_p\mathbb{R}^3 \simeq T_pM \oplus \nu_p.$$

Let $\mathrm{pr}_p \colon T_p\mathbb{R}^3 \to T_pM$ be the projection to the tangent space of M at p.

If X is a C^∞ vector field along M, then $X_p \in T_p\mathbb{R}^3$. Define a vector field $\mathrm{pr}(X)$ on M by

$$\mathrm{pr}(X)_p = \mathrm{pr}_p(X_p) \in T_pM.$$

Proposition 6.8. *If X is a C^∞ vector field along M in $\mathbb{R}^3$, then the vector field* $\mathrm{pr}(X)$ *on M defined above is C^∞.*

Proof. For each $p \in M$, there is a neighborhood U of p in M on which there is a C^∞ unit normal vector field N (by Section 5.1). For $q \in U$,

$$\mathrm{pr}_q(X_q) = X_q - \langle X_q, N_q \rangle N_q.$$

Note that pr_q does not depend on the choice of the unit normal vector field; $-N_q$ would have given the same answer. As q varies in U,

$$\mathrm{pr}(X) = X - \langle X, N \rangle N,$$

which shows that $\mathrm{pr}(X)$ is C^∞ on U. Hence, $\mathrm{pr}(X)$ is C^∞ at p. Since p is an arbitrary point of M, the vector field $\mathrm{pr}(X)$ is C^∞ on M. □

According to this proposition, the projection operator is a map

$$\mathrm{pr}\colon \Gamma(T\mathbb{R}^3|_M) \to \mathfrak{X}(M).$$

6.5 The Riemannian Connection on a Surface in $\mathbb{R}^3$

We continue to consider a smooth, not necessarily orientable surface M in $\mathbb{R}^3$. Given two C^∞ vector fields X and Y on M and a point $p \in M$, the directional derivative $D_{X_p}Y$ is not in general tangent to M. Define

$$(\nabla_X Y)_p := \nabla_{X_p} Y = \mathrm{pr}_p(D_{X_p}Y) \in T_pM, \tag{6.9}$$

where $\mathrm{pr}_p\colon T_p\mathbb{R}^3 \to T_pM$ is the orthogonal projection defined in Section 6.4. As p varies over M,

$$\nabla_X Y = \mathrm{pr}(D_X Y),$$

which, by Proposition 6.8, shows that $\nabla_X Y$ is a C^∞ vector field on M. So we have a map

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M).$$

Proposition 6.9. *The Riemannian connection on a smooth, not necessarily orientable surface M in $\mathbb{R}^3$ is given by* $\nabla_X Y = \mathrm{pr}(D_X Y)$.

Proof. Problem 6.6. □

Problems

6.1. Convex linear combination of connections
Show that if ∇ and ∇' are connections on a manifold, then their sum $\nabla + \nabla'$ is *not* a connection. However, a convex linear combination $t\nabla + (1-t)\nabla'$ for any $t \in \mathbb{R}$ is a connection. More generally, show that if $\nabla^1, \dots, \nabla^n$ are connections on a manifold, then the linear combination $\sum_{i=1}^n a_i \nabla^i$ is a connection provided $\sum_{i=1}^n a_i = 1$.

6.2. $\mathcal{F}$-linearity of the torsion
Prove that the torsion $T(X,Y)$ is $\mathcal{F}$-linear in X and in Y.

6.3. $\mathcal{F}$-linearity of the curvature
Prove that the curvature $R(X,Y)Z$ is $\mathcal{F}$-linear in Z.

6.4. Existence of a Riemannian connection
Prove that formula (6.8) defines a torsion-free affine connection ∇ that is compatible with the metric.

6.5. Orthogonal projection
Let M be a smooth surface in $\mathbb{R}^3$. Show that the orthogonal projection $\mathrm{pr}\colon \Gamma(T\mathbb{R}^3|_M) \to \mathfrak{X}(M)$ is $\mathcal{F}$-linear.

6.6. Riemannian connection on a surface in $\mathbb{R}^3$
Prove Proposition 6.9. □

6.7. Lie derivative
On any manifold M, let $\mathcal{L}\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $\mathcal{L}(X,Y) = \mathcal{L}_X Y$, be the Lie derivative. Show that $\mathcal{L}_X Y$ satisfies the Leibniz rule in Y, but is not $\mathcal{F}$-linear in X. (*Hint*: $\mathcal{L}_X Y = [X,Y]$.)

6.8. Riemannian connection of a submanifold
Let $S \subset M$ be an immersed submanifold of a Riemannian manifold M and let ∇ be the Riemannian connection on M. Show that the Riemannian connection ∇' on S is given by

$$\nabla'_X Y = (\nabla_X Y)_{\tan}.$$

§7 Vector Bundles

The set $\mathfrak{X}(M)$ of all C^∞ vector fields on a manifold M has the structure of a real vector space, which is the same as a module over the field $\mathbb{R}$ of real numbers. Let $\mathcal{F} = C^\infty(M)$ again be the ring of C^∞ functions on M. Since we can multiply a vector field by a C^∞ function, the vector space $\mathfrak{X}(M)$ is also a module over $\mathcal{F}$. Thus the set $\mathfrak{X}(M)$ has two module structures, over $\mathbb{R}$ and over $\mathcal{F}$. In speaking of a linear map: $\mathfrak{X}(M) \to \mathfrak{X}(M)$ one should be careful to specify whether it is $\mathbb{R}$-linear or $\mathcal{F}$-linear. An $\mathcal{F}$-linear map is of course $\mathbb{R}$-linear, but the converse is not true.

The $\mathcal{F}$-linearity of the torsion $T(X,Y)$ and the curvature $R(X,Y)Z$ from the preceding section has an important consequence, namely that these two constructions make sense pointwise. For example, if X_p, Y_p, and Z_p are tangent vectors to a manifold M at p, then one can define $R(X_p,Y_p)Z_p$ to be $(R(X,Y)Z)_p \in T_pM$ for *any* vector fields X, Y, and Z on M that extend X_p, Y_p, and Z_p, respectively. While it is possible to explain this fact strictly within the framework of vector fields, it is most natural to study it in the context of vector bundles. For this reason, we make a digression on vector bundles in this section.

We will try to understand $\mathcal{F}$-linear maps from the point of view of vector bundles. The main result (Theorem 7.26) asserts the existence of a one-to-one correspondence between $\mathcal{F}$-linear maps $\alpha\colon \Gamma(E) \to \Gamma(F)$ of sections of vector bundles and bundle maps $\varphi\colon E \to F$.

7.1 Definition of a Vector Bundle

Given an open subset U of a manifold M, one can think of $U \times \mathbb{R}^r$ as a family of vector spaces $\mathbb{R}^r$ parametrized by the points in U. A ***vector bundle***, intuitively speaking, is a family of vector spaces that locally "looks" like $U \times \mathbb{R}^r$.

Definition 7.1. A C^∞ surjection $\pi\colon E \to M$ is a C^∞ ***vector bundle of rank*** r if

(i) For every $p \in M$, the set $E_p := \pi^{-1}(p)$ is a real vector space of dimension r;
(ii) every point $p \in M$ has an open neighborhood U such that there is a fiber-preserving diffeomorphism

$$\phi_U\colon \pi^{-1}(U) \to U \times \mathbb{R}^r$$

that restricts to a linear isomorphism $E_p \to \{p\} \times \mathbb{R}^r$ on each fiber.

The space E is called the ***total space***, the space M the ***base space***, and the space E_p the ***fiber above*** p of the vector bundle. We often say that E is a ***vector bundle over*** M. A vector bundle of rank 1 is also called a ***line bundle***.

Condition (i) says that $\pi\colon E \to M$ is a family of vector spaces, while condition (ii) formalizes the fact that this family is locally looks like $\mathbb{R}^n$. We call the open set U in (ii) a ***trivializing open subset*** for the vector bundle, and ϕ_U a ***trivialization*** of

$\pi^{-1}(U)$. A ***trivializing open cover*** for the vector bundle is an open cover $\{U_\alpha\}$ of M consisting of trivializing open sets U_α together with trivializations $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$.

Example 7.2 (*Product bundle*). If V is a vector space of dimension r, then the projection $\pi : M \times V \to M$ is a vector bundle of rank r, called a ***product bundle***. Via the projection $\pi : S^1 \times \mathbb{R} \to S^1$, the cylinder $S^1 \times \mathbb{R}$ is a product bundle over the circle S^1.

Example 7.3 (*Möbius strip*). The open Möbius strip is the quotient of $[0,1] \times \mathbb{R}$ by the identification

$$(0,t) \sim (1,-t).$$

It is a vector bundle of rank 1 over the circle (Figure 7.1).

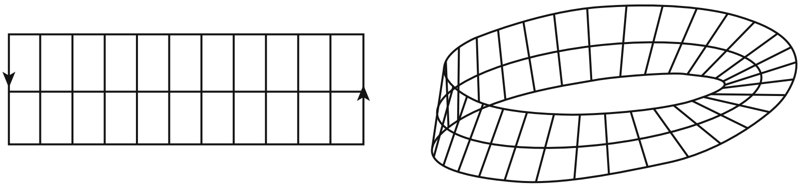


Fig. 7.1. A Möbius strip

Example 7.4 (*Restriction of a vector bundle*). Let S be a submanifold of a manifold M, and $\pi : E \to M$ a C^∞ vector bundle. Then $\pi_S : \pi^{-1}(S) \to S$ is also a vector bundle, called the ***restriction*** of E to S, written $E|_S := \pi^{-1}(S)$ (Figure 7.2).

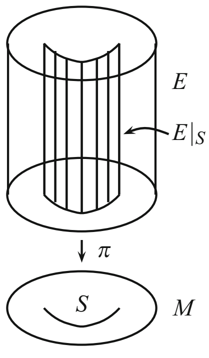


Fig. 7.2. A vector bundle on M restricted to S

Definition 7.5. Let $\pi_E : E \to M$ and $\pi_F : F \to N$ be C^∞ vector bundles. A C^∞ ***bundle map*** from E to F is a pair of C^∞ maps $(\varphi\colon E \to F, \underline{\varphi}\colon M \to N)$ such that

(i) the diagram

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ M & \xrightarrow[\underline{\varphi}]{} & N, \end{array}$$

commutes,

(ii) φ restricts to a linear map $\varphi_p\colon E_p \to F_{\underline{\varphi}(p)}$ of fibers for each $p \in M$.

Abusing language, we often call the map $\varphi\colon E \to F$ alone the bundle map.

An important special case of a bundle map occurs when E and F are vector bundles over the same manifold M and the base map $\underline{\varphi}$ is the identity map $\mathbb{1}_M$. In this case we call the bundle map $(\varphi\colon E \to F, \mathbb{1}_M)$ a bundle map ***over*** M.

If there is a bundle map $\psi\colon F \to E$ over M such that $\psi \circ \varphi = \mathbb{1}_E$ and $\varphi \circ \psi = \mathbb{1}_F$, then φ is called a ***bundle isomorphism over*** M, and the vector bundles E and F are said to be ***isomorphic over*** M.

Definition 7.6. A vector bundle $\pi\colon E \to M$ is said to be ***trivial*** if it is isomorphic to a product bundle $M \times \mathbb{R}^r \to M$ over M.

Example 7.7 (*Tangent bundle*). For any manifold M, define TM to be the set of all tangent vectors of M:

$$TM = \{(p,v) \mid p \in M,\ v \in T_pM\}.$$

If U is a coordinate open subset of M, then TU is bijective with the product bundle $U \times \mathbb{R}^n$. We give TM the topology generated by TU as U runs over all coordinate open subsets of M. In this way TM can be given a manifold structure so that $TM \to M$ becomes a vector bundle. It is called the ***tangent bundle*** of M (for details, see [21, Section 12]).

Example 7.8. If $f\colon M \to N$ is a C^∞ map of manifolds, then its differential gives rise to a bundle map $f_*\colon TM \to TN$ defined by

$$f_*(p,v) = (f(p), f_{*,p}(v)).$$

7.2 The Vector Space of Sections

A ***section*** of a vector bundle $\pi\colon E \to M$ over an open set U is a function $s\colon U \to E$ such that $\pi \circ s = \mathbb{1}_U$, the identity map on U. For each $p \in U$, the section s picks out one element of the fiber E_p. The set of all C^∞ sections of E over U is denoted by $\Gamma(U,E)$. If U is the manifold M, we also write $\Gamma(E)$ instead of $\Gamma(M,E)$.

The set $\Gamma(U,E)$ of C^∞ sections of E over U is clearly a vector space over $\mathbb{R}$. It is in fact a module over the ring $C^\infty(U)$ of C^∞ functions, for if f is a C^∞ function over U and s is a C^∞ section of E over U, then the definition

$$(fs)(p) := f(p)s(p) \in E_p, \quad p \in U,$$

makes fs into a C^∞ section of E over U.

Example 7.9 (*Sections of a product line bundle*). A section s of the product bundle $M \times \mathbb{R} \to M$ is a map $s(p) = (p, f(p))$. So there is a one-to-one correspondence

$$\{\text{sections of } M \times \mathbb{R} \to M\} \longleftrightarrow \{\text{functions } f\colon M \to \mathbb{R}\}.$$

In particular, the space of C^∞ sections of the product line bundle $M \times \mathbb{R} \to M$ may be identified with $C^\infty(M)$.

Example 7.10 (*Sections of the tangent bundle*). A vector field on a manifold M assigns to each point $p \in M$ a tangent vector $X_p \in T_pM$. Therefore, it is precisely a section of the tangent bundle TM. Thus, $\mathfrak{X}(M) = \Gamma(TM)$.

Example 7.11 (*Vector fields along a submanifold*). If M is a regular submanifold of $\mathbb{R}^n$, then a C^∞ vector field along M is precisely a section of the restriction $T\mathbb{R}^n|_M$ of $T\mathbb{R}^n$ to M. This explains our earlier notation $\Gamma(T\mathbb{R}^3|_M)$ for the space of C^∞ vector fields along M in $\mathbb{R}^3$.

Definition 7.12. A bundle map $\varphi\colon E \to F$ over a manifold M (meaning that the base map is the identity $\mathbb{1}_M$) induces a map on the space of sections:

$$\varphi_\#\colon \Gamma(E) \to \Gamma(F),$$
$$\varphi_\#(s) = \varphi \circ s.$$

This induced map $\varphi_\#$ is $\mathcal{F}$-linear because if $f \in \mathcal{F}$, then

$$\begin{aligned}
\big(\varphi_\#(fs)\big)(p) &= \big(\varphi \circ (fs)\big)(p) = \varphi\big(f(p)s(p)\big) \\
&= f(p)\varphi\big(s(p)\big) \qquad \text{(because } \varphi \text{ is } \mathbb{R}\text{-linear on each fiber)} \\
&= \big(f\varphi_\#(s)\big)(p).
\end{aligned}$$

Our goal in the rest of this chapter is to prove that conversely, every $\mathcal{F}$-linear map $\alpha\colon \Gamma(E) \to \Gamma(F)$ comes from a bundle map $\varphi\colon E \to F$, i.e., $\alpha = \varphi_\#$.

7.3 Extending a Local Section to a Global Section

Consider the interval $(-\pi/2, \pi/2)$ as an open subset of the real line $\mathbb{R}$. The example of the tangent function $\tan\colon (-\pi/2, \pi/2) \to \mathbb{R}$ shows that it may not be possible to extend the domain of a C^∞ function $f\colon U \to \mathbb{R}$ from an open subset $U \subset M$ to the manifold M. However, given a point $p \in U$, it is always possible to find a C^∞ global function $\bar{f}\colon M \to \mathbb{R}$ that agrees with f on *some* neighborhood of p. More generally, this is also true for sections of a vector bundle.

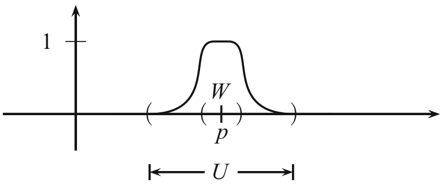


Fig. 7.3. A bump function supported in U

Proposition 7.13. *Let $E \to M$ be a C^∞ vector bundle, s a C^∞ section of E over some open set U in M, and p a point in U. Then there exists a C^∞ global section $\bar{s} \in \Gamma(M,E)$ that agrees with s over some neighborhood of p.*

Proof. Choose a C^∞ bump function f on M such that $f \equiv 1$ on a neighborhood W of p contained in U and $\operatorname{supp} f \subset U$ (Figure 7.3). Define $\bar{s}\colon M \to E$ by

$$\bar{s}(q) = \begin{cases} f(q)s(q) & \text{for } q \in U, \\ 0 & \text{for } q \notin U. \end{cases}$$

On U the section $\bar{s}$ is clearly C^∞ for it is the product of a C^∞ function f and a C^∞ section s.

If $p \notin U$, then $p \notin \operatorname{supp} f$. Since $\operatorname{supp} f$ is a closed set, there is a neighborhood V of p contained in its complement $M \setminus \operatorname{supp} f$. On V the section $\bar{s}$ is identically zero. Hence, $\bar{s}$ is C^∞ at p. This proves that $\bar{s}$ is C^∞ on M.

On W, since $f \equiv 1$, the section $\bar{s}$ agrees with s. □

7.4 Local Operators

In this section, E and F are C^∞ vector bundles over a manifold M, and $\mathcal{F}$ is the ring $C^\infty(M)$ of C^∞ functions on M.

Definition 7.14. Let E and F be vector bundles over a manifold M. An $\mathbb{R}$-linear map $\alpha\colon \Gamma(E) \to \Gamma(F)$ is a ***local operator*** if whenever a section $s \in \Gamma(E)$ vanishes on an open set U in M, then $\alpha(s) \in \Gamma(F)$ also vanishes on U. It is a ***point operator*** if whenever a section $s \in \Gamma(E)$ vanishes at a point p in M, then $\alpha(s) \in \Gamma(F)$ also vanishes at p.

Example 7.15. By Example 7.9, the vector space $C^\infty(\mathbb{R})$ of C^∞ functions on $\mathbb{R}$ may be identified with the vector space $\Gamma(\mathbb{R} \times \mathbb{R})$ of C^∞ sections of the product line bundle over $\mathbb{R}$. The derivative

$$\frac{d}{dt}\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$$

is a local operator since if $f(t) \equiv 0$ on U, then $f'(t) \equiv 0$ on U. However, d/dt is not a point operator.

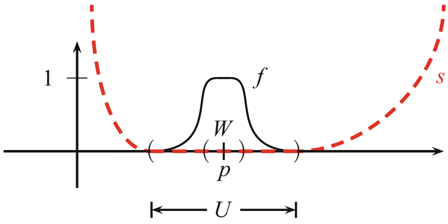


Fig. 7.4. The product fs of a bump function f and a section s is zero.

Example 7.16. Let $\Omega^k(M)$ denote the vector space of C^∞ k-forms on a manifold M. Then the exterior derivative $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ is a local operator.

Proposition 7.17. *Let E and F be C^∞ vector bundles over a manifold M, and $\mathcal{F} = C^\infty(M)$. If a map $\alpha\colon \Gamma(E) \to \Gamma(F)$ is $\mathcal{F}$-linear, then it is a local operator.*

Proof. Suppose the section $s \in \Gamma(E)$ vanishes on the open set U. Let $p \in U$ and let f be a C^∞ bump function such that $f(p) = 1$ and $\operatorname{supp} f \subset U$ (Figure 7.3). Then $fs \in \Gamma(E)$ and $fs \equiv 0$ on M (Figure 7.4). So $\alpha(fs) \equiv 0$. By $\mathcal{F}$-linearity,

$$0 = \alpha(fs) = f\alpha(s).$$

Evaluating at p gives $\alpha(s)(p) = 0$. Since p is an arbitrary point of U, $\alpha(s) \equiv 0$ on U. □

Example 7.18. On a C^∞ manifold M, a derivation $D\colon C^\infty(M) \to C^\infty(M)$ is $\mathbb{R}$-linear, but not $\mathcal{F}$-linear since by the Leibniz rule,

$$D(fg) = (Df)g + fDg, \quad \text{for } f, g \in \mathcal{F}.$$

However, by Problem 7.1, D is a local operator.

Example 7.19. Fix a C^∞ vector field $X \in \mathfrak{X}(M)$. Then a connection ∇ on M induces a map

$$\nabla_X\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$$

that satisfies the Leibniz rule. By Problem 7.2, ∇_X is a local operator.

7.5 Restriction of a Local Operator to an Open Subset

A continuous global section of a vector bundle can always be restricted to an open subset, but in general a section over an open subset cannot be extended to a continuous global section. For example, the tangent function defined on the open interval $(-\pi/2, \pi/2)$ cannot be extended to a continuous function on the real line. Nonetheless, a local operator, which is defined on global sections of a vector bundle, can always be restricted to an open subset.

Theorem 7.20. *Let E and F be vector bundles over a manifold M. If $\alpha\colon \Gamma(E) \to \Gamma(F)$ is a local operator, then for each open subset U of M there is a unique linear map, called the* ***restriction*** *of α to U,*

$$\alpha_U : \Gamma(U,E) \to \Gamma(U,F)$$

such that for any global section $t \in \Gamma(E)$,

$$\alpha_U(t|_U) = \alpha(t)|_U. \tag{7.1}$$

Proof. Let $s \in \Gamma(U,E)$ and $p \in U$. By Proposition 7.13, there exists a global section $\bar{s}$ of E that agrees with s in some neighborhood W of p in U. We define $\alpha_U(s)(p)$ using (7.1):

$$\alpha_U(s)(p) = \alpha(\bar{s})(p).$$

Suppose $\tilde{s} \in \Gamma(E)$ is another global section that agrees with s in the neighborhood W of p. Then $\bar{s} = \tilde{s}$ in W. Since α is a local operator, $\alpha(\bar{s}) = \alpha(\tilde{s})$ on W. Hence, $\alpha(\bar{s})(p) = \alpha(\tilde{s})(p)$. This shows that $\alpha_U(s)(p)$ is independent of the choice of $\bar{s}$, so α_U is well defined and unique. Fix $p \in U$. If $s \in \Gamma(U,E)$ and $\bar{s} \in \Gamma(M,E)$ agree on a neighborhood W of p, then $\alpha_U(s) = \alpha(\bar{s})$ on W. Hence, $\alpha_U(s)$ is C^∞ as a section of F.

If $t \in \Gamma(M,E)$ is a global section, then it is a global extension of its restriction $t|_U$. Hence,

$$\alpha_U(t|_U)(p) = \alpha(t)(p)$$

for all $p \in U$. This proves that $\alpha_U(t|_U) = \alpha(t)|_U$. □

Proposition 7.21. *Let E and F be C^∞ vector bundles over a manifold M, let U be an open subset of M, and let $\mathcal{F}(U) = C^\infty(U)$, the ring of C^∞ functions on U. If $\alpha\colon \Gamma(E) \to \Gamma(F)$ is $\mathcal{F}$-linear, then the restriction $\alpha_U : \Gamma(U,E) \to \Gamma(U,F)$ is $\mathcal{F}(U)$-linear.*

Proof. Let $s \in \Gamma(U,E)$ and $f \in \mathcal{F}(U)$. Fix $p \in U$ and let $\bar{s}$ and $\bar{f}$ be global extensions of s and f that agree with s and f, respectively, on a neighborhood of p (Proposition 7.13). Then

$$\begin{aligned}
\alpha_U(fs)(p) &= \alpha(\bar{f}\bar{s})(p) && \text{(definition of } \alpha_U\text{)}\\
&= \bar{f}(p)\alpha(\bar{s})(p) && \text{(}\mathcal{F}\text{-linearity of } \alpha\text{)}\\
&= f(p)\alpha_U(s)(p).
\end{aligned}$$

Since p is an arbitrary point of U,

$$\alpha_U(fs) = f\alpha_U(s),$$

proving that α_U is $\mathcal{F}(U)$-linear. □

7.6 Frames

A ***frame*** for a vector bundle E of rank r over an open set U is a collection of sections $e_1,\dots,e_r$ of E over U such that at each point $p \in U$, the elements $e_1(p),\dots,e_r(p)$ form a basis for the fiber E_p.

Proposition 7.22. *A C^∞ vector bundle $\pi\colon E \to M$ is trivial if and only if it has a C^∞ frame.*

Proof. Suppose E is trivial, with C^∞ trivialization

$$\phi\colon E \to M \times \mathbb{R}^r.$$

Let $v_1,\dots,v_r$ be the standard basis for $\mathbb{R}^r$. Then the elements (p,v_i), $i = 1,\dots,r$, form a basis for $\{p\} \times \mathbb{R}^r$ for each $p \in M$, and so the sections of E

$$e_i(p) = \phi^{-1}(p,v_i), \quad i = 1,\dots,r,$$

provide a basis for E_p at each point $p \in M$.

Conversely, suppose $e_1,\dots,e_r$ is a frame for $E \to M$. Then every point $e \in E$ is a linear combination $e = \sum a^i e_i$. The map

$$\phi(e) = (\pi(e),a^1,\dots,a^r)\colon E \to M \times \mathbb{R}^r$$

is a bundle map with inverse

$$\psi\colon M \times \mathbb{R}^r \to E,$$
$$\psi(p,a^1,\dots,a^r) = \sum a^i(p)e_i(p). \qquad \square$$

It follows from this proposition that over any trivializing open set U of a vector bundle E, there is always a frame.

7.7 $\mathcal{F}$-Linearity and Bundle Maps

Throughout this subsection, E and F are C^∞ vector bundles over a manifold M, and $\mathcal{F} = C^\infty(M)$ is the ring of C^∞ real-valued functions on M. We will show that an $\mathcal{F}$-linear map $\alpha\colon \Gamma(E) \to \Gamma(F)$ can be defined pointwise and therefore corresponds uniquely to a bundle map $E \to F$.

Lemma 7.23. *An $\mathcal{F}$-linear map $\alpha\colon \Gamma(E) \to \Gamma(F)$ is a point operator.*

Proof. We need to show that if $s \in \Gamma(E)$ vanishes at a point p in M, then $\alpha(s) \in \Gamma(F)$ also vanishes at p. Let U be an open neighborhood of p over which E is trivial. Thus, over U it is possible to find a frame $e_1,\dots,e_r$ for E. We write

$$s|_U = \sum a^i e_i, \quad a^i \in C^\infty(U) = \mathcal{F}(U).$$

Because s vanishes at p, all $a^i(p) = 0$. Since α is $\mathcal{F}$-linear, it is a local operator (Proposition 7.17) and by Theorem 7.20 its restriction $\alpha_U\colon \Gamma(U,E) \to \Gamma(U,F)$ is defined. Then

$$\begin{aligned}
\alpha(s)(p) &= \alpha_U(s|_U)(p) && \text{(Theorem 7.20)}\\
&= \alpha_U\left(\sum a^i e_i\right)(p) \\
&= \left(\sum a^i \alpha_U(e_i)\right)(p) && (\alpha_U \text{ is } \mathcal{F}(U)\text{-linear (Proposition 7.21))}\\
&= \sum a^i(p)\alpha_U(e_i)(p) = 0.
\end{aligned}$$

□

Lemma 7.24. *Let E and F be C^∞ vector bundles over a manifold M. A fiber-preserving map $\varphi\colon E \to F$ that is linear on each fiber is C^∞ if and only if $\varphi_\#$ takes C^∞ sections of E to C^∞ sections of F.*

Proof. ($\Rightarrow$) If φ is C^∞, then $\varphi_\#(s) = \varphi \circ s$ clearly takes a C^∞ section s of E to a C^∞ section of F.
($\Leftarrow$) Fix $p \in M$ and let $(U, x^1, \dots, x^n)$ be a chart about p over which E and F are both trivial. Let $e_1, \dots, e_r \in \Gamma(E)$ be a frame for E over U. Likewise, let $f_1, \dots, f_m \in \Gamma(F)$ be a frame for F over U. A point of $E|_U$ can be written as a unique linear combination $\sum a^j e_j$. Suppose

$$\varphi \circ e_j = \sum_i b_j^i f_i.$$

In this expression the b_j^i's are C^∞ functions on U, because by hypothesis $\varphi \circ e_j = \varphi_\#(e_j)$ is a C^∞ section of F. Then

$$\varphi \circ \left(\sum_j a^j e_j\right) = \sum_{i,j} a^j b_j^i f_i.$$

One can take local coordinates on $E|_U$ to be $(x^1, \dots, x^n, a^1, \dots, a^r)$. In terms of these local coordinates,

$$\varphi(x^1, \dots, x^n, a^1, \dots, a^r) = \left(x^1, \dots, x^n, \sum_j a^j b_j^1, \dots, \sum_j a^j b_j^m\right),$$

which is a C^∞ map. □

Proposition 7.25. *If $\alpha\colon \Gamma(E) \to \Gamma(F)$ is $\mathcal{F}$-linear, then for each $p \in M$, there is a unique linear map $\varphi_p\colon E_p \to F_p$ such that for all $s \in \Gamma(E)$,*

$$\varphi_p\big(s(p)\big) = \alpha(s)(p).$$

Proof. Given $e \in E_p$, to define $\varphi_p(e)$, choose any section $s \in \Gamma(E)$ such that $s(p) = e$ (Problem 7.4) and define

$$\varphi_p(e) = \alpha(s)(p) \in F_p.$$

This definition is independent of the choice of the section s, because if s' is another section of E with $s'(p) = e$, then $(s - s')(p) = 0$ and so by Lemma 7.23, we have $\alpha(s - s')(p) = 0$, i.e.,

$$\alpha(s)(p) = \alpha(s')(p).$$

Let us show that $\varphi_p\colon E_p \to F_p$ is linear. Suppose $e_1, e_2 \in E_p$ and $a_1, a_2 \in \mathbb{R}$. Let s_1, s_2 be global sections of E such that $s_i(p) = e_i$. Then $(a_1 s_1 + a_2 s_2)(p) = a_1 e_1 + a_2 e_2$, so

$$\begin{aligned}\varphi_p(a_1e_1 + a_2e_2) &= \alpha(a_1s_1 + a_2s_2)(p)\\ &= a_1\alpha(s_1)(p) + a_2\alpha(s_2)(p)\\ &= a_1\varphi_p(e_1) + a_2\varphi_p(e_2).\end{aligned}$$ □

Theorem 7.26. *There is a one-to-one correspondence*

$$\{\text{bundle maps } \varphi\colon E \to F\} \longleftrightarrow \{\mathcal{F}\text{-linear maps } \alpha\colon \Gamma(E) \to \Gamma(F)\},$$
$$\varphi \longmapsto \varphi_\# \,.$$

Proof. We first show surjectivity. Suppose $\alpha\colon \Gamma(E) \to \Gamma(F)$ is $\mathcal{F}$-linear. By the preceding proposition, for each $p \in M$ there is a linear map $\varphi_p\colon E_p \to F_p$ such that for any $s \in \Gamma(E)$,

$$\varphi_p(s(p)) = \alpha(s)(p).$$

Define $\varphi\colon E \to F$ by $\varphi(e) = \varphi_p(e)$ if $e \in E_p$.

For any $s \in \Gamma(E)$ and for every $p \in M$,

$$\big(\varphi_\#(s)\big)(p) = \varphi\big(s(p)\big) = \alpha(s)(p),$$

which shows that $\alpha = \varphi_\#$. Since $\varphi_\#$ takes C^∞ sections of E to C^∞ sections of F, by Lemma 7.24 the map $\varphi\colon E \to F$ is C^∞. Thus, φ is a bundle map.

Next we prove the injectivity of the correspondence. Suppose $\varphi,\ \psi\colon E \to F$ are two bundle maps such that $\varphi_\# = \psi_\#\colon \Gamma(E) \to \Gamma(F)$. For any $e \in E_p$, choose a section $s \in \Gamma(E)$ such that $s(p) = e$. Then

$$\begin{aligned}\varphi(e) = \varphi\big(s(p)\big) &= \big(\varphi_\#(s)\big)(p) = \big(\psi_\#(s)\big)(p)\\ &= (\psi \circ s)(p) = \psi(e).\end{aligned}$$

Hence, $\varphi = \psi$. □

Corollary 7.27. *An $\mathcal{F}$-linear map $\omega\colon \mathfrak{X}(M) \to C^\infty(M)$ is a C^∞ 1-form on M.*

Proof. By Proposition 7.25, one can define for each $p \in M$, a linear map $\omega_p\colon T_pM \to \mathbb{R}$ such that for all $X \in \mathfrak{X}(M)$,

$$\omega_p(X_p) = \omega(X)(p).$$

This shows that ω is a 1-form on M.

For every C^∞ vector field X on M, $\omega(X)$ is a C^∞ function on M. This shows that as a 1-form, ω is C^∞. □

7.8 Multilinear Maps over Smooth Functions

By Proposition 7.25, if $\alpha\colon \Gamma(E) \to \Gamma(F)$ is an $\mathcal{F}$-linear map of sections of vector bundles over M, then at each $p \in M$, it is possible to define a linear map $\varphi_p\colon E_p \to F_p$ such that for any $s \in \Gamma(E)$,

$$\varphi_p(s(p)) = \alpha(s)(p).$$

This can be generalized to $\mathcal{F}$-multilinear maps.

Proposition 7.28. *Let E, E', F be vector bundles over a manifold M. If*

$$\alpha\colon \Gamma(E) \times \Gamma(E') \to \Gamma(F)$$

is $\mathcal{F}$-bilinear, then for each $p \in M$ there is a unique $\mathbb{R}$-bilinear map

$$\varphi_p\colon E_p \times E'_p \to F_p$$

such that for all $s \in \Gamma(E)$ and $s' \in \Gamma(E')$,

$$\varphi_p(s(p), s'(p)) = (\alpha(s,s'))(p).$$

Since the proof is similar to that of Proposition 7.25, we leave it as an exercise.

Of course, Proposition 7.28 generalizes to $\mathcal{F}$-linear maps with k arguments. Just as in Corollary 7.27, we conclude that if an alternating map

$$\omega\colon \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) (k \text{ times}) \to C^\infty(M)$$

is $\mathcal{F}$-linear in each argument, then ω induces a k-form $\tilde{\omega}$ on M such that for $X_1, \dots, X_k \in \mathfrak{X}(M)$,

$$\tilde{\omega}_p(X_{1,p}, \dots, X_{k,p}) = (\omega(X_1, \dots, X_k))(p).$$

It is customary to write the k-form $\tilde{\omega}$ as ω.

Problems

7.1. Derivations are local operators
Let M be a smooth manifold. Show that a derivation $D\colon C^\infty(M) \to C^\infty(M)$ is a local operator.

7.2. The Leibniz rule and local operators
Let $\pi\colon E \to M$ be a vector bundle and let $X \in \mathfrak{X}(M)$. A map $\alpha_X\colon \Gamma(E) \to \Gamma(E)$ satisfies the ***Leibniz rule*** if for any $f \in C^\infty(M)$ and $s \in \Gamma(E)$,

$$\alpha_X(fs) = (Xf)s + f\alpha_X(s).$$

Such a map α_X is of course not $\mathcal{F}$-linear, but show that it is a local operator.

7.3. The Lie bracket and the Leibniz rule
Let X be a smooth vector field on a smooth manifold M. Define a linear map $\alpha_X\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $\alpha_X(Y) = [X,Y]$. Show that α_X satisfies the Leibniz rule.

7.4. Section with a prescribed value
Suppose $\pi\colon E \to M$ is a C^∞ vector bundle. Let $p \in M$ and $e \in E_p$. Show that E has a C^∞ section s with $s(p) = e$.

7.5. Coefficients relative to a frame
Suppose the vector bundle $\pi\colon E \to M$ has a C^∞ global frame $e_1,\dots,e_k$. Then every point v in E can be written uniquely in the form $\sum a^j(v)e_j$. Prove that the functions $a^j\colon E \to \mathbb{R}$ are C^∞.

7.6. $\mathcal{F}$-bilinear maps
Prove Proposition 7.28.

§8 Gauss's Theorema Egregium

For a surface in $\mathbb{R}^3$ we defined its Gaussian curvature K at a point p by taking normal sections of the surface, finding the maximum κ_1 and the minimum κ_2 of the curvature of the normal sections, and setting K to be the product of κ_1 and κ_2. So defined, the Gaussian curvature evidently depends on how the surface is isometrically embedded in $\mathbb{R}^3$.

On the other hand, an abstract Riemannian manifold has a unique Riemannian connection. The curvature tensor $R(X,Y)$ of the Riemannian connection is then completely determined by the Riemannian metric and so is an intrinsic invariant of the Riemannian manifold, independent of any embedding. We think of a surface in $\mathbb{R}^3$ as a particular isometric embedding of an abstract Riemannian manifold of dimension 2. For example, both a plane and a cylinder are locally isometric embeddings of the same abstract surface, as one sees by simply bending a piece of paper. We will show in this chapter that the Gaussian curvature of a surface in $\mathbb{R}^3$ is expressible in terms of the curvature tensor $R(X,Y)$ and the metric. Hence, it too depends only on the metric, not on the particular embedding into $\mathbb{R}^3$.

8.1 The Gauss and Codazzi–Mainardi Equations

Suppose M is a regular submanifold of $\mathbb{R}^n$. For X a tangent vector field on M and Z a vector field *along* M in $\mathbb{R}^n$, we defined in Section 4.5 the directional derivative $D_X Z$. It is a vector field along M in $\mathbb{R}^n$.

For $X,Y \in \mathfrak{X}(M)$ and $Z \in \Gamma(T\mathbb{R}^n|_M)$, we verified in Proposition 4.10 the equation

$$R(X,Y)Z := D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z = 0. \tag{8.1}$$

Note that this is not the same curvature operator as the curvature operator of the directional derivative on $\mathbb{R}^n$. The earlier curvature operator was a map

$$R\colon \mathfrak{X}(\mathbb{R}^n) \times \mathfrak{X}(\mathbb{R}^n) \times \mathfrak{X}(\mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n);$$

the current curvature operator is a map

$$R\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(T\mathbb{R}^n|M) \to \mathfrak{X}(T\mathbb{R}^n|M).$$

Since the Riemannian connection ∇ of a surface M in $\mathbb{R}^3$ is defined in terms of the directional derivative D on M, it is easy to compare the curvature tensors of ∇ and D. This will lead to a formula for the curvature R of ∇, called the ***Gauss curvature equation***.

A vector field $Y \in \mathfrak{X}(M)$ is also a vector field *along* M in $\mathbb{R}^3$. Hence, if $X,Y \in \mathfrak{X}(M)$, the directional derivative $D_X Y$ makes sense. Assume that M is oriented with a unit normal vector field N. At any point $p \in M$, the normal component of the vector $D_{X_p} Y$ is $\langle D_{X_p} Y, N_p \rangle N_p$, and therefore the tangent component is

$$\mathrm{pr}(D_{X_p}Y) = D_{X_p}Y - \langle D_{X_p}Y, N_p\rangle N_p.$$

By (6.9), this is the definition of the Riemannian connection $\nabla_{X_p}Y$ on M. Hence,

$$\begin{aligned} D_{X_p}Y &= \nabla_{X_p}Y + \langle D_{X_p}Y, N_p\rangle N_p \\ &= \nabla_{X_p}Y + \langle L_p(X_p), Y_p\rangle N_p \quad \text{(by Lemma 5.2).} \end{aligned} \tag{8.2}$$

Globalizing this equation, we have, for $X, Y \in \mathfrak{X}(M)$,

$$D_X Y = \nabla_X Y + \langle L(X), Y\rangle N, \tag{8.3}$$

which decomposes the vector field $D_X Y$ into its tangential and normal components.

Theorem 8.1. *Let M be an oriented surface in $\mathbb{R}^3$, ∇ its Riemannian connection, R the curvature operator of ∇, and L the shape operator on M. For $X, Y, Z \in \mathfrak{X}(M)$,*

(i) (Gauss curvature equation)

$$R(X,Y)Z = \langle L(Y), Z\rangle L(X) - \langle L(X), Z\rangle L(Y);$$

(ii) (Codazzi–Mainardi equation)

$$\nabla_X L(Y) - \nabla_Y L(X) - L([X,Y]) = 0.$$

Proof. (i) Decomposing $D_Y Z$ into its tangential and normal components, one has

$$\begin{aligned} D_Y Z &= (D_Y Z)_{\mathrm{tan}} + (D_Y Z)_{\mathrm{nor}} \\ &= \nabla_Y Z + \langle L(Y), Z\rangle N. \end{aligned} \tag{8.4}$$

Hence,

$$\begin{aligned} D_X D_Y Z &= D_X \nabla_Y Z + D_X\big(\langle L(Y), Z\rangle N\big) \\ &= \nabla_X \nabla_Y Z + \langle L(X), \nabla_Y Z) N + X\langle L(Y), Z\rangle N + \langle L(Y), Z\rangle D_X N. \\ &\qquad \text{(by (8.4) and Leibniz rule)} \\ &= \nabla_X \nabla_Y Z - \langle L(X), Z\rangle L(X) + \langle L(X), \nabla_Y Z\rangle N + X\langle L(Y), Z\rangle N. \end{aligned} \tag{8.5}$$

Interchanging X and Y gives

$$D_Y D_x Z = \nabla_Y \nabla_X Z - \langle L(X), Z\rangle L(Y) + \langle L(Y), \nabla_X Z\rangle N + Y\langle L(X), Z\rangle N. \tag{8.6}$$

By (8.4),

$$D_{[X,Y]}Z = \nabla_{[X,Y]}Z + \langle L([X,Y]), Z\rangle N. \tag{8.7}$$

Subtracting (8.6) and (8.7) from (8.5) gives

$$\begin{aligned} 0 = R_D(X,Y)Z = R(X,Y)Z - \langle L(Y), Z\rangle L(X) + \langle L(X), Z\rangle L(Y) \\ + \text{normal component.} \end{aligned} \tag{8.8}$$

The tangential component of (8.8) gives

$$R(X,Y) = \langle L(Y), Z\rangle L(X) - \langle L(X), Z\rangle L(Y),$$

which is the Gauss curvature equation.

(ii) The normal component of (8.8) is

$$\big(\langle L(X), \nabla_Y Z\rangle + X\langle L(Y), Z\rangle \\ - \langle L(Y), \nabla_X Z\rangle - Y\langle L(X), Z\rangle - \langle L([X,Y]), Z\rangle\big)N = 0.$$

By the compatibility of ∇ with the metric, this simplifies to

$$\langle \nabla_X L(Y), Z\rangle - \langle \nabla_Y L(X), Z\rangle - \langle L([X,Y]), Z\rangle = 0.$$

Since the equation above is true for all Z, by the nondegeneracy of the inner product,

$$\nabla_X L(Y) - \nabla_Y L(X) - L([X,Y]) = 0. \qquad \square$$

Remark 8.2. By the generalization of Proposition 7.28 to $\mathcal{F}$-trilinear maps, because $R(X,Y)Z$ is $\mathcal{F}$-trilinear in X, Y, and Z, for each $p \in M$, there is a unique $\mathbb{R}$-trilinear map

$$R_p\colon T_pM \times T_pM \times T_pM \to T_pM$$

such that $R_p(X_p, Y_p)Z_p = (R(X,Y)Z)_p$. Thus, although we have stated the Gauss curvature equation for vector fields, it is also true for vectors $X_p, Y_p, Z_p \in T_pM$ at a point p.

8.2 A Proof of the Theorema Egregium

Suppose for every Riemannian manifold M, there is defined a function $f_M\colon M \to \mathbb{R}$. The function f_M is said to be an ***isometric invariant*** if for every isometry $\varphi\colon M \to \tilde{M}$ of Riemannian manifolds, we have $f_M(p) = f_{\tilde{M}}(\varphi(p))$ for all $p \in M$.

Theorem 8.3 (Theorema Egregium). *Let M be a surface in $\mathbb{R}^3$ and p a point in M.*

(i) *If e_1, e_2 is an orthonormal basis for the tangent plane T_pM, then the Gaussian curvature K_p of M at p is*

$$K_p = \langle R_p(e_1, e_2)e_2, e_1\rangle. \tag{8.9}$$

(ii) *The Gaussian curvature is an isometric invariant of smooth surfaces in $\mathbb{R}^3$.*

Proof. (i) In Corollary 5.7 we found a formula for the Gaussian curvature K_p in terms of the shape operator L and an orthonormal basis e_1, e_2 for T_pM:

$$K_p = \langle L(e_1), e_1\rangle\langle L(e_2), e_2\rangle - \langle L(e_1), e_2\rangle\langle L(e_2), e_1\rangle.$$

By the Gauss curvature equation,

$$R_p(e_1, e_2)e_2 = \langle L(e_2), e_2\rangle L(e_1) - \langle L(e_1), e_2\rangle L(e_2).$$

Taking the inner product with e_1 gives

$$\begin{aligned}\langle R_p(e_1, e_2)e_2, e_1\rangle &= \langle L(e_2), e_2\rangle\langle L(e_1), e_1\rangle - \langle L(e_1), e_2\rangle\langle L(e_2), e_1\rangle \\ &= K_p.\end{aligned}$$

(ii) Since $R_p(e_1, e_2)$ is determined completely by the metric, by Formula (8.9) the same can be said of K_p. A detailed proof is left to Problem 8.3. $\square$

Gauss's Theorema Egregium has some practical implications. For example, since a sphere and a plane have different Gaussian curvatures, it is not possible to map any open subset of the sphere isometrically to an open subset of the plane. Since the earth is spherical, this proves the impossibility of making a truly accurate map of any region of the earth, no matter how small.

8.3 The Gaussian Curvature in Terms of an Arbitrary Basis

The Theorema Egregium gives a formula for the Gaussian curvature of a surface in terms of an orthonormal basis for the tangent plane at a point. From it, one can derive a formula for the Gaussian curvature in terms of an arbitrary basis.

Proposition 8.4. *Let M be a smooth surface in $\mathbb{R}^3$ and $p \in M$. If u, v is any basis for the tangent plane T_pM, then the Gaussian curvature at p is*

$$K_p = \frac{\langle R_p(u,v)v, u\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}.$$

Proof. Let e_1, e_2 be an orthonormal basis for T_pM and suppose

$$u = ae_1 + be_2,$$
$$v = ce_1 + de_2.$$

By the skew-symmetry of the curvature tensor,

$$R_p(u,v) = (ad-bc)R_p(e_1,e_2),$$

so that

$$\langle R_p(u,v)v,u\rangle = (ad-bc)^2\langle R_p(e_1,e_2)e_2,e_1\rangle.$$

On the other hand,

$$\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2$$
$$= (a^2+b^2)(c^2+d^2) - (ac+bd)^2 = (ad-bc)^2.$$

Hence,

$$K_p = \langle R_p(e_1,e_2)e_2,e_1\rangle = \frac{\langle R_p(u,v)v,u\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}.$$ □

Problems

8.1. Affine connection under a diffeomorphism
A diffeomorphism $\varphi\colon M \to \tilde{M}$ of smooth manifolds induces an isomorphism

$$\varphi_*\colon \mathfrak{X}(M) \to \mathfrak{X}(\tilde{M})$$

of their Lie algebras of vector fields. (In particular, $\varphi_*([X,Y]) = [\varphi_*X, \varphi_*Y]$.) Suppose $\tilde{M}$ has an affine connection $\tilde{\nabla}$. For $X, Y \in \mathfrak{X}(M)$, define the vector field $\nabla_X Y \in \mathfrak{X}(M)$ by

$$\varphi_*(\nabla_X Y) = \tilde{\nabla}_{\varphi_* X}(\varphi_* Y).$$

(a) Show that for all $f \in C^\infty(M)$, $\varphi_*(fX) = (f \circ \varphi^{-1})\varphi_* X$.
(b) Show that $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is an affine connection on M.

8.2. Riemannian connection and curvature under an isometry

(a) Now suppose that $\varphi\colon M \to \tilde{M}$ is an isometry of Riemannian manifolds. Show that if $\tilde{\nabla}$ is the Riemannian connection on $\tilde{M}$, then the connection defined in Problem 8.1 is the Riemannian connection on M.
(b) Let R and $\tilde{R}$ be the curvature tensors of ∇ and $\tilde{\nabla}$, respectively. Show that for all $X, Y, Z \in \mathfrak{X}(M)$,

$$\varphi_*(R(X,Y)Z) = \tilde{R}(\varphi_* X, \varphi_* Y)\varphi_* Z.$$

8.3. Gaussian curvature is an isometric invariant
Suppose $\varphi\colon M \to \tilde{M}$ is an isometry of smooth surfaces in $\mathbb{R}^3$. Show that the Gaussian curvature of M at a point $p \in M$ is equal to the Gaussian curvature $\tilde{K}$ of $\tilde{M}$ at $\varphi(p)$: $K_p = \tilde{K}_{\varphi(p)}$.

§9 Generalizations to Hypersurfaces in $\mathbb{R}^{n+1}$

Much of what we have done for surfaces in $\mathbb{R}^3$ in the preceding sections generalizes readily to higher dimensions. We now carry this out.

9.1 The Shape Operator of a Hypersurface

A ***hypersurface*** M in $\mathbb{R}^{n+1}$ is a submanifold of codimension 1. Although all of our hypersurfaces are by definition smooth, sometimes for emphasis we adopt the redundant locution "smooth hypersurface." Assume that there is a smooth unit normal vector field N on M; note that this is always possible locally on any hypersurface. By Proposition 4.3 we know that the directional derivative D on $\mathbb{R}^{n+1}$ is the Riemannian connection of $\mathbb{R}^{n+1}$.

For any point $p \in M$ and tangent vector $X_p \in T_pM$, since $\langle N,N\rangle \equiv 1$,

$$0 = X_p\langle N,N\rangle = 2\langle D_{X_p}N,N\rangle.$$

Therefore, $D_{X_p}N$ is tangent to M. The ***shape operator*** $L_p\colon T_pM \to T_pM$ is defined to be

$$L_p(X_p) = -D_{X_p}N \quad \text{for } X_p \in T_pM.$$

Recall that $\mathfrak{X}(M)$ is the vector space of C^∞ vector fields and $\mathcal{F} = C^\infty(M)$ the ring of C^∞ functions on M. As the point p varies over M, the shape operator globalizes to an $\mathcal{F}$-linear map $L\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by $L(X)_p = L_p(X_p)$.

Proposition 9.1. *Let $X,Y \in \mathfrak{X}(M)$ be C^∞ vector fields on an oriented hypersurface M in $\mathbb{R}^{n+1}$. Then*

(i) $\langle L(X),Y\rangle = \langle D_XY,N\rangle$.
(ii) *the shape operator is self-adjoint with respect to the Euclidean metric inherited from $\mathbb{R}^{n+1}$:*

$$\langle L(X),Y\rangle = \langle X,L(Y)\rangle.$$

Proof. Since $\langle Y,N\rangle = 0$, by the compatibility of D with the metric,

$$0 = X\langle Y,N\rangle = \langle D_XY,N\rangle + \langle Y,D_XN\rangle.$$

Hence,

$$\langle D_XY,N\rangle = \langle Y,-D_XN\rangle = \langle Y,L(X)\rangle. \tag{9.1}$$

Reversing the roles of X and Y gives

$$\langle D_YX,N\rangle = \langle X,L(Y)\rangle. \tag{9.2}$$

Since $[X,Y]$ is tangent to M,

$$\langle [X,Y],N\rangle = 0. \tag{9.3}$$

By torsion-freeness, subtracting (9.2) and (9.3) from (9.1) gives

$$0 = \langle D_XY - D_YX - [X,Y],N\rangle = \langle Y,L(X)\rangle - \langle X,L(Y)\rangle. \qquad \square$$

Since the shape operator $L_p\colon T_pM \to T_pM$ is self-adjoint, all of its eigenvalues $\lambda_1,\dots,\lambda_n$ are real. These are the ***principal curvatures*** of the hypersurface M at p. We define the ***mean curvature*** of M at p to be the average of the principal curvatures:

$$H(p) = \frac{1}{n}\sum_{i=1}^{n}\lambda_i = \frac{1}{n}\operatorname{tr}(L_p),$$

and the ***Gaussian curvature*** at p to be the determinant of the shape operator:

$$K(p) = \prod_{i=1}^{n}\lambda_i = \det(L_p).$$

9.2 The Riemannian Connection of a Hypersurface

We learned earlier that the unique Riemannian connection of $\mathbb{R}^{n+1}$ is the directional derivative D.

Theorem 9.2. *Let M be a smooth hypersurface in $\mathbb{R}^{n+1}$ and D the directional derivative on $\mathbb{R}^{n+1}$. For $X,Y \in \mathfrak{X}(M)$, the tangential component of D_XY defines the Riemannian connection ∇ of M:*

$$\nabla_X Y = (D_XY)_{\tan}.$$

Proof. Since it is evident that ∇ satisfies the two defining properties of a connection, it suffices to check that $\nabla_X Y$ is torsion-free and compatible with the metric.

1) Torsion-freeness: Let T_D and T_∇ be the torsions of D and ∇, respectively. By definition, for $X,Y \in \mathfrak{X}(M)$,

$$D_XY = \nabla_XY + (D_XY)_{\mathrm{nor}}, \tag{9.4}$$
$$D_YX = \nabla_YX + (D_YX)_{\mathrm{nor}}. \tag{9.5}$$

Since D is torsion-free,

$$D_XY - D_YX = [X,Y].$$

Equating the normal components of both sides, we get $(D_XY)_{\mathrm{nor}} - (D_YX)_{\mathrm{nor}} = 0$. Therefore, by (9.4) and (9.5),

$$\begin{aligned} 0 = T_D(X,Y) &= D_XY - D_YX - [X,Y] \\ &= \big(\nabla_XY - \nabla_YX - [X,Y]\big) + (D_XY)_{\mathrm{nor}} - (D_YX)_{\mathrm{nor}} \\ &= \nabla_XY - \nabla_YX - [X,Y] = T_\nabla(X,Y). \end{aligned}$$

2) Compatibility with the metric: For $X,Y,Z \in \mathfrak{X}(M)$,

$$\begin{aligned} X\langle Y,Z\rangle &= \langle D_XY,Z\rangle + \langle Y,D_XZ\rangle \\ &= \langle \nabla_XY + (D_XY)_{\mathrm{nor}},Z\rangle + \langle Y,\nabla_XZ + (\nabla_XY)_{\mathrm{nor}}\rangle \\ &= \langle \nabla_XY,Z\rangle + \langle Y,\nabla_XZ\rangle. \end{aligned}$$

Therefore, ∇_XY is the Riemannian connection on E. □

9.3 The Second Fundamental Form

At each point p of an oriented hypersurface $M \subset \mathbb{R}^{n+1}$, there is a sequence of symmetric bilinear maps on T_pM, called the ***first***, ***second***, and ***third fundamental forms***, and so on:

$$\begin{aligned}
\mathrm{I}(X_p,Y_p) &= \langle X_p,Y_p\rangle,\\
\mathrm{II}(X_p,Y_p) &= \langle L(X_p),Y_p\rangle = \langle X_p,L(Y_p)\rangle,\\
\mathrm{III}(X_p,Y_p) &= \langle L^2(X_p),Y_p\rangle = \langle L(X_p),L(Y_p)\rangle = \langle X_p,L^2(Y_p)\rangle,\dots.
\end{aligned}$$

For $X,Y \in \mathfrak{X}(M)$, the directional derivative D_XY decomposes into the sum of a tangential component and a normal component. The tangential component $(D_XY)_{\mathrm{tan}}$ is the Riemannian connection on M; the normal component $(D_XY)_{\mathrm{nor}}$ is essentially the second fundamental form.

Proposition 9.3. *If N is a smooth unit normal vector field on the hypersurface M in $\mathbb{R}^{n+1}$ and $X,Y \in \mathfrak{X}(M)$ are smooth vector fields on M, then*

$$(D_XY)_{\mathrm{nor}} = \mathrm{II}(X,Y)N.$$

Proof. The normal component $(D_XY)_{\mathrm{nor}}$ is a multiple of N, so

$$\begin{aligned}
D_XY &= (D_XY)_{\mathrm{tan}} + (D_XY)_{\mathrm{nor}}\\
&= \nabla_XY + \lambda N
\end{aligned}$$

for some $\lambda \in \mathbb{R}$. Taking the inner product of both sides with N gives

$$\langle D_XY,N\rangle = \lambda\langle N,N\rangle = \lambda.$$

Hence,

$$\begin{aligned}
(D_XY)_{\mathrm{nor}} &= \langle D_XY,N\rangle N\\
&= \langle L(X),Y\rangle N\\
&= \mathrm{II(X,Y)N}. \qquad \square
\end{aligned}$$

9.4 The Gauss Curvature and Codazzi–Mainardi Equations

Let M be an oriented, smooth hypersurface in $\mathbb{R}^{n+1}$, with a smooth unit normal vector field N. The relation between the Riemannian connection D on $\mathbb{R}^{n+1}$ and the Riemannian connection on M implies a relation between the curvature $R_D(X,Y)Z = 0$ on $\mathbb{R}^{n+1}$ and the curvature $R(X,Y)Z$ on M. The tangential component of this relation gives the Gauss curvature equation, and the normal component gives the Codazzi-Mainardi equation.

Theorem 9.4. *If M is an oriented, smooth hypersurface in $\mathbb{R}^{n+1}$ and $X,Y,Z \in \mathfrak{X}(M)$, then*

(i) *(the Gauss curvature equation)*

$$R(X,Y) = \langle L(Y), Z\rangle L(X) - \langle L(X), Z\rangle L(Y).$$

(ii) *(the Codazzi–Mainardi equation)*

$$\nabla_X L(Y) - \nabla_Y L(X) - L([X,Y]) = 0.$$

Proof. The proof is identical to that of Theorem 8.1. □

Corollary 9.5. *If M is a smooth oriented hypersurface in $\mathbb{R}^{n+1}$ with curvature R and $X, Y \in \mathfrak{X}(M)$, then*

$$\langle R(X,Y)Y, X\rangle = \mathrm{II}(X,X)\,\mathrm{II}(Y,Y) - \mathrm{II}(X,Y)^2.$$

Proof. By the Gauss curvature equation,

$$\begin{aligned}\langle R(X,Y)Y, X\rangle &= \langle L(Y), Y\rangle\langle L(X), X\rangle - \langle L(X), Y\rangle\langle L(Y), X\rangle \\ &= \mathrm{II}(X,X)\,\mathrm{II}(Y,Y) - \mathrm{II}(X,Y)^2.\end{aligned}$$ □

When M is a surface in $\mathbb{R}^3$ and X_p and Y_p form an orthonormal basis for T_pM, the quantity on the right of the corollary is the Gaussian curvature of the surface at p (see Corollary 5.7(ii) and Theorem 8.3(i)).

Chapter 2

Curvature and Differential Forms

In Chapter 1 we developed, in terms of vector fields, the classical theory of curvature for curves and surfaces in $\mathbb{R}^3$. There is a dual approach using differential forms. Differential forms arise naturally even if one is interested only in vector fields. For example, the coefficients of tangent vectors relative to a frame on an open set are differential 1-forms on the open set. Differential forms are more supple than vector fields: they can be differentiated and multiplied, and they behave functorially under the pullback by a smooth map. In the 1920s and 30s Élie Cartan pioneered the use of differential forms in differential geometry [4], and these have proven to be tools of great power and versatility.

In this chapter, we redevelop the theory of connections and curvature in terms of differential forms. First, in Section 10 we generalize the notion of a connection from the tangent bundle to an arbitrary vector bundle. Then in Section 11 we show how to represent connections, curvature, and torsion by differential forms. Finally, in a demonstration of their utility, in Section 12 we use differential forms to reprove Gauss's Theorema Egregium.

§10 Connections on a Vector Bundle

The affine connection generalizes the directional derivative from $\mathbb{R}^n$ to an arbitrary manifold, but does not include the case of the directional derivative of a vector field along a submanifold of $\mathbb{R}^n$. For that, we need the notion of a connection on a vector bundle. An affine connection on a manifold is simply a connection on the tangent bundle. Because of the asymmetry of the two arguments in a connection $\nabla_X s$ on a vector bundle, the torsion is no longer defined, but the curvature still makes sense.

We define a Riemannian metric on a vector bundle, so that a Riemannian metric on a manifold becomes a Riemannian metric on the tangent bundle. Compatibility of a connection with the metric again makes sense. However, the lack of a concept

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of torsion means that it is no longer possible to single out a unique connection on a vector bundle such as the Riemannian connection on a Riemannian manifold.

A connection on a vector bundle turns out to be a local operator, and like all local operators, it restricts to any open subset (Theorem 7.20).

10.1 Connections on a Vector Bundle

Definition 10.1. Let $E \to M$ be a C^∞ vector bundle over M. A ***connection*** on E is a map

$$\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$

such that for $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$,

(i) $\nabla_X s$ is $\mathcal{F}$-linear in X and $\mathbb{R}$-linear in s;
(ii) (Leibniz rule) if f is a C^∞ function on M, then

$$\nabla_X(fs) = (Xf)s + f\nabla_X s.$$

Since $Xf = (df)(X)$, the Leibniz rule may be written as

$$\nabla_X(fs) = (df)(X)s + f\nabla_X s,$$

or, suppressing X,

$$\nabla(fs) = df \cdot s + f\nabla s.$$

Example. An affine connection on a manifold M is a connection on the tangent bundle $TM \to M$.

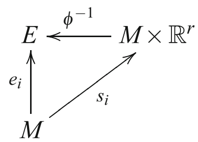
Example 10.2. Let M be a submanifold of $\mathbb{R}^n$ and $E = T\mathbb{R}^n|_M$ the restriction of the tangent bundle of $\mathbb{R}^n$ to M. In Section 4.5, we defined the directional derivative

$$D\colon \mathfrak{X}(M) \times \Gamma(T\mathbb{R}^n|_M) \to \Gamma(T\mathbb{R}^n|_M).$$

By Proposition 4.9, it is a connection on the vector bundle $T\mathbb{R}^n|_M$.

We say that a section $s \in \Gamma(E)$ is ***flat*** if $\nabla_X s = 0$ for all $X \in \mathfrak{X}(M)$.

Example 10.3 (*Induced connection on a trivial bundle*). Let E be a trivial bundle of rank r over a manifold M. Thus, there is a bundle isomorphism $\phi\colon E \to M \times \mathbb{R}^r$, called a ***trivialization*** for E over M. The trivialization $\phi\colon E \xrightarrow{\sim} M \times \mathbb{R}^r$ induces a connection on E as follows. If $v_1, \ldots, v_r$ is a basis for $\mathbb{R}^r$, then $s_i\colon p \mapsto (p, v_i)$, $i = 1, \ldots, r$, define a global frame for the product bundle $M \times \mathbb{R}^r$ over M, and $e_i = \phi^{-1} \circ s_i$, $i = 1, \ldots, r$, define a global frame for E over M:



So every section $s \in \Gamma(E)$ can be written uniquely as a linear combination

$$s = \sum h^i e_i, \quad h^i \in \mathcal{F}.$$

We can define a connection ∇ on E by declaring the sections e_i to be flat and applying the Leibniz rule and $\mathbb{R}$-linearity to define $\nabla_X s$:

$$\nabla_X s = \nabla_X \left(\sum h^i e_i \right) = \sum (X h^i) e_i. \tag{10.1}$$

Exercise 10.4. Check that (10.1) defines a connection on the trivial bundle E.

The connection ∇ on a trivial bundle induced by a trivialization depends on the trivialization, for the flat sections for ∇ are precisely the sections of E corresponding to the constant sections of $M \times \mathbb{R}^r$ under the trivialization.

10.2 Existence of a Connection on a Vector Bundle

In Section 10.1 we defined a connection on a vector bundle and exhibited a connection on a trivial bundle. We will now show the existence of a connection on an arbitrary vector bundle.

Let ∇^0 and ∇^1 be two connections on a vector bundle E over M. By the Leibniz rule, for any vector field $X \in \mathfrak{X}(M)$, section $s \in \Gamma(E)$, and function $f \in C^\infty(M)$,

$$\nabla^0_X (fs) = (Xf)s + f \nabla^0_X s, \tag{10.2}$$

$$\nabla^1_X (fs) = (Xf)s + f \nabla^1_X s. \tag{10.3}$$

Hence,

$$(\nabla^0_X + \nabla^1_X)(fs) = 2(Xf)s + f(\nabla^0_X + \nabla^1_X)s. \tag{10.4}$$

Because of the extra factor of 2 in (10.4) the sum of two connections does not satisfy the Leibniz rule and so is not a connection. However, if we multiply (10.2) by $1-t$ and (10.3) by t, then $(1-t)\nabla^0_X + t\nabla^1_X$ satisfies the Leibniz rule. More generally, the same idea can be used to prove the following proposition.

Proposition 10.5. *Any finite linear combination $\sum t_i \nabla^i$ of connections ∇^i is a connection provided the coefficients add up to* 1*, that is, $\sum t_i = 1$.*

A finite linear combination whose coefficients add up to 1 is called a ***convex linear combination***. Using Proposition 10.5, we now prove the existence of a connection.

Theorem 10.6. *Every C^∞ vector bundle E over a manifold M has a connection.*

Proof. Fix a trivializing open cover $\{U_\alpha\}$ for E and a partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$. On each U_α, the vector bundle $E|_{U_\alpha}$ is trivial and so has a connection ∇^α by Example 10.3. For $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$, denote by s_α the restriction of s to U_α and define

$$\nabla_X s = \sum \rho_\alpha \nabla^\alpha_X s_\alpha.$$

Although this looks like an infinite sum, it is in fact a finite sum in a neighborhood of each point p in M, for by the local finiteness of $\{\operatorname{supp}\rho_\alpha\}$, there is an open neighborhood U of p that intersects only finitely many of the sets $\operatorname{supp}\rho_\alpha$. This means that on U all except finitely many of the ρ_α's are zero and so $\sum \rho_\alpha \nabla^\alpha_X s_\alpha$ is a finite sum on U with $\sum \rho_\alpha = 1$.

By checking on the open set U, it is easy to show that $\nabla_X s$ is $\mathcal{F}$-linear in X, is $\mathbb{R}$-linear in s, and satisfies the Leibniz rule. Hence, ∇ is a connection on E. □

Note the similarity of this proof to the proof for the existence of a Riemannian metric on a vector bundle (Theorem 10.8). For a linear combination of connections to be a connection, we require the coefficients to sum to 1. On the other hand, for a linear combination of Riemannian metrics to be a Riemannian metric, the requirement is that all the coefficients be nonnegative with at least one coefficient positive.

10.3 Curvature of a Connection on a Vector Bundle

The concept of torsion does not make sense for a connection on an arbitrary vector bundle, but curvature still does. It is defined by the same formula as for an affine connection: for $X, Y \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$,

$$R(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \in \Gamma(E).$$

So R is an $\mathbb{R}$-multilinear map

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E).$$

As before, $R(X,Y)s$ is $\mathcal{F}$-linear in all three arguments and so it is actually defined pointwise. Moreover, because $R_p(X_p, Y_p)$ is skew-symmetric in X_p and Y_p, at every point p there is an alternating bilinear map

$$R_p\colon T_pM \times T_pM \to \operatorname{Hom}(E_p, E_p) =: \operatorname{End}(E_p)$$

into the endomorphism ring of E_p. We call this map the ***curvature tensor*** of the connection ∇.

10.4 Riemannian Bundles

We can also generalize the notion of a Riemannian metric to vector bundles. Let $E \to M$ be a C^∞ vector bundle over a manifold M. A ***Riemannian metric*** on E assigns to each $p \in M$ an inner product $\langle\ ,\ \rangle_p$ on the fiber E_p; the assignment is required to be C^∞ in the following sense: if s and t are C^∞ sections of E, then $\langle s, t\rangle$ is a C^∞ function on M.

Thus, a Riemannian metric on a manifold M is simply a Riemannian metric on the tangent bundle TM. A vector bundle together with a Riemannian metric is called a ***Riemannian bundle***.

Example 10.7. Let E be a trivial vector bundle of rank r over a manifold M, with trivialization $\phi\colon E \xrightarrow{\sim} M \times \mathbb{R}^r$. The Euclidean inner product $\langle\ ,\ \rangle_{\mathbb{R}^r}$ on $\mathbb{R}^r$ induces a Riemannian metric on E via the trivialization ϕ: if $u, v \in E_p$, then the fiber map $\phi_p\colon E_p \to \mathbb{R}^r$ is a linear isomorphism and we define

$$\langle u, v\rangle = \langle \phi_p(u), \phi_p(v)\rangle_{\mathbb{R}^r}.$$

It is easy to check that $\langle\ ,\ \rangle$ is a Riemannian metric on E.

The proof of Theorem 1.12 generalizes to prove the existence of a Riemannian metric on a vector bundle.

Theorem 10.8. *On any C^∞ vector bundle $\pi\colon E \to M$, there is a Riemannian metric.*

Proof. Let $\{U_\alpha, \varphi_\alpha\colon E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^k\}$ be a trivializing open cover for E. On $E|_{U_\alpha}$ there is a Riemannian metric $\langle\ ,\ \rangle_\alpha$ induced from the trivial bundle $U_\alpha \times \mathbb{R}^k$. Let $\{\rho_\alpha\}$ be a partition of unity on M subordinate to the open cover $\{U_\alpha\}$. By the same reasoning as in Theorem 1.12, the sum

$$\langle\ ,\ \rangle := \sum \rho_\alpha \langle\ ,\ \rangle_\alpha$$

is a finite sum in a neighborhood of each point of M and is a Riemannian metric on E. □

10.5 Metric Connections

We say that a connection ∇ on a Riemannian bundle E is ***compatible with the metric*** if for all $X \in \mathfrak{X}(M)$ and $s, t \in \Gamma(E)$,

$$X\langle s, t\rangle = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle.$$

A connection compatible with the metric on a Riemannian bundle is also called a ***metric connection***.

Example. Let E be a trivial vector bundle of rank r over a manifold M, with trivialization $\phi\colon E \xrightarrow{\sim} M \times \mathbb{R}^r$. We showed in Example 10.3 that the trivialization induces a connection ∇ on E and in Example 10.7 that the trivialization induces a Riemannian metric $\langle\ ,\ \rangle$ on E.

Proposition 10.9. *On a trivial vector bundle E over a manifold M with trivialization $\phi\colon E \xrightarrow{\sim} M \times \mathbb{R}^r$, the connection ∇ on E induced by the trivialization ϕ is compatible with the Riemannian metric $\langle\ ,\ \rangle$ on E induced by the trivialization.*

Proof. Let $v_1, \ldots, v_r$ be an orthonormal basis for $\mathbb{R}^r$ and $e_1, \ldots, e_r$ the corresponding global frame for E, where $e_i(p) = \phi^{-1}(p, v_i)$. Then $e_1, \ldots, e_r$ is an orthonormal flat frame for E with respect to the Riemannian metric and the connection on E induced by ϕ. If $s = \sum a^i e_i$ and $t = \sum b^j e_j$ are C^∞ sections of E, then

$$\begin{aligned} X\langle s, t\rangle &= X\left(\sum a^i b^i\right) && \text{(because } e_1, \ldots, e_r \text{ are orthonormal)} \\ &= \sum (Xa^i) b^i + \sum a^i X b^i \\ &= \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle && \text{(see (10.1)).} \end{aligned}$$

□

Example 10.10 (*Connection on $T\mathbb{R}^n|_M$*). If M is a submanifold of $\mathbb{R}^n$, the Euclidean metric on $\mathbb{R}^n$ restricts to a Riemannian metric on the vector bundle $T\mathbb{R}^n|_M$. Sections of the vector bundle $T\mathbb{R}^n|_M$ are vector fields along M in $\mathbb{R}^n$. As noted in Example 10.2, the directional derivative in $\mathbb{R}^n$ induces a connection

$$D\colon \mathfrak{X}(M) \times \Gamma(T\mathbb{R}^n|_M) \to \Gamma(T\mathbb{R}^n|_M).$$

Proposition 4.10 asserts that the connection D on $T\mathbb{R}^n|_M$ has zero curvature and is compatible with the metric.

The Gauss curvature equation for a surface M in $\mathbb{R}^3$, a key ingredient of the proof of Gauss's Theorema Egregium, is a consequence of the vanishing of the curvature tensor of the connection D on the bundle $T\mathbb{R}^3|_M$ (Theorem 8.1).

Lemma 10.11. *Let $E \to M$ be a Riemannian bundle. Suppose $\nabla^1, \dots, \nabla^m$ are connections on E compatible with the metric and $a_1, \dots, a_m$ are C^∞ functions on M that add up to* 1. *Then $\nabla = \sum_i a_i \nabla^i$ is a connection on E compatible with the metric.*

Proof. By Proposition 10.5, ∇ is a connection on E. It remains to check that ∇ is compatible with the metric. If $X \in \mathfrak{X}(M)$ and $s,t \in \Gamma(E)$, then

$$X\langle s,t\rangle = \langle \nabla^i_X s, t\rangle + \langle s, \nabla^i_X t\rangle \tag{10.5}$$

for all i because ∇^i is compatible with the metric. Now multiply (10.5) by a_i and sum:

$$\begin{aligned} X\langle s,t\rangle &= \sum a_i X\langle s,t\rangle \\ &= \left\langle \sum a_i \nabla^i_X s, t\right\rangle + \left\langle s, \sum a_i \nabla^i_X t\right\rangle \\ &= \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle. \end{aligned}$$ □

Proposition 10.12. *On any Riemannian bundle $E \to M$, there is a connection compatible with the metric.*

Proof. Choose a trivializing open cover $\{U_\alpha\}$ for E. By Proposition 10.9, for each trivializing open set U_α of the cover, we can find a connection ∇^α on $E|_{U_\alpha}$ compatible with the metric. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to the open cover $\{U_\alpha\}$. Because the collection $\{\operatorname{supp}\rho_\alpha\}$ is locally finite, the sum $\nabla = \sum \rho_\alpha \nabla^\alpha$ is a finite sum in a neighborhood of each point. Since $\sum \rho_\alpha \equiv 1$, by Lemma 10.11, ∇ is a connection on E compatible with the metric. □

10.6 Restricting a Connection to an Open Subset

A connection ∇ on a vector bundle E over M,

$$\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$

is $\mathcal{F}$-linear in the first argument, but not $\mathcal{F}$-linear in the second argument. However, it turns out that the $\mathcal{F}$-linearity in the first argument and the Leibniz rule in the second argument are enough to imply that ∇ is a local operator.

Proposition 10.13. *Let ∇ be a connection on a vector bundle E over a manifold M, X be a smooth vector field on M, and s a smooth section of E. If either X or s vanishes identically on an open subset U, then $\nabla_X s$ vanishes identically on U.* (see Problem 7.2.)

Proof. Suppose X vanishes identically on U. Since $\nabla_X s$ is $\mathcal{F}$-linear in X, by Proposition 7.17, for any $s \in \Gamma(E)$, $\nabla_X s$ vanishes identically on U.

Next, suppose $s \equiv 0$ on U and $p \in U$. Choose a C^∞ bump function f such that $f \equiv 1$ in a neighborhood of p and $\operatorname{supp} f \subset U$. By our choice of f, the derivative $X_p f$ is zero. Since s vanishes on the support of f, we have $fs \equiv 0$ on M and so $\nabla_X(fs) \equiv 0$. Evaluating at p gives

$$0 = (\nabla_X(fs))_p = (Xf)_p s_p + f(p)(\nabla_X s)_p = (\nabla_X s)_p.$$

Because p is an arbitrary point of U, we conclude that $\nabla_X s$ vanishes identically on U. □

In the same way that a local operator: $\Gamma(E) \to \Gamma(F)$ can be restricted to any open subset (Theorem 7.20), a connection on a vector bundle can be restricted to any open subset. More precisely, given a connection ∇ on a vector bundle E, for every open set U there is a connection

$$\nabla^U : \mathfrak{X}(U) \times \Gamma(U,E) \to \Gamma(U,E)$$

such that for any global vector field $\bar{X} \in \mathfrak{X}(M)$ and global section $\bar{s} \in \Gamma(E)$,

$$\nabla^U_{\bar{X}|_U}(\bar{s}|_U) = (\nabla_{\bar{X}}\bar{s})|_U.$$

Suppose $X \in \mathfrak{X}(U)$ and $s \in \Gamma(U,E)$. For any $p \in U$, to define $\nabla^U_X s \in \Gamma(U,E)$ first pick a global vector field $\bar{X}$ and a global section $\bar{s} \in \Gamma(E)$ that agree with X and s in a neighborhood of p. Then define

$$(\nabla^U_X s)(p) = (\nabla_{\bar{X}}\bar{s})(p). \tag{10.6}$$

Because $\nabla_{\bar{X}}\bar{s}$ is a local operator in $\bar{X}$ and in $\bar{s}$, this definition is independent of the choice of $\bar{X}$ and $\bar{s}$. It is a routine matter to show that ∇^U satisfies all the properties of a connection on $E|_U$ (Problem 10.1).

10.7 Connections at a Point

Suppose ∇ is a connection on a vector bundle E over a manifold M. For $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$, since $\nabla_X s$ is $\mathcal{F}$-linear in X, it is a point operator in X and Proposition 7.25 assures us that it can be defined pointwise in X: there is a unique map, also denoted by ∇,

$$\nabla : T_pM \times \Gamma(E) \to E_p$$

such that if $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$, then

$$\nabla_{X_p} s = (\nabla_X s)_p.$$

It is easy to check that $\nabla_{X_p} s$ has the following properties: for $X_p \in T_pM$ and $s \in \Gamma(E)$,

(i) $\nabla_{X_p} s$ is $\mathbb{R}$-linear in X_p and in s;
(ii) if f is a C^∞ function on M, then

$$\nabla_{X_p}(fs) = (X_p f)s(p) + f(p)\nabla_{X_p} s.$$

Problems

10.1. Restriction of a connection to an open subset
Show that the restriction of a connection to an open subset U given by (10.6) defines a connection on $E|_U$.

10.2. Restriction of a Riemannian connection
Let U be an open subset of a Riemannian manifold M. Prove that if ∇ is the Riemannian connection on M, then the restriction ∇^U is the Riemannian connection on U.

10.3. Agreement on a curve
Given a connection ∇ on a vector bundle $E \to M$, a point p in M, and a tangent vector X_p in T_pM, show that if two sections s and t of E agree on a curve through p in M with initial vector X_p, then $\nabla_{X_p} s = \nabla_{X_p} t$.

§11 Connection, Curvature, and Torsion Forms

According to Gauss's Theorema Egregium, if R_p is the curvature tensor at a point p of a surface M in $\mathbb{R}^3$ and u, v is any orthonormal frame for the tangent plane T_pM, then the Gaussian curvature of the surface at p is

$$K_p = \langle R_p(u,v)v, u\rangle. \tag{11.1}$$

Since this formula does not depend on the embedding of the surface in $\mathbb{R}^3$, but only on the Riemannian structure of the surface, it makes sense for an abstract Riemannian 2-manifold and can be taken as the definition of the Gaussian curvature at a point of such a surface, for example, the hyperbolic upper half-plane $\mathbb{H}^2$ (Problem 1.7). To compute the Gaussian curvature from (11.1), one would need to compute first the Riemannian connection ∇ using the six-term formula (6.8) and then compute the curvature tensor $R_p(u,v)v$—a clearly nontrivial task.

One of the great advantages of differential forms is its computability, and so in this section we shall recast connections, curvature, and torsion in terms of differential forms. This will lead to a simple computation of the Gaussian curvature of the hyperbolic upper half-plane in the next section.

Relative to a frame for a vector bundle, a connection on the bundle can be represented by a matrix of 1-forms, and the curvature by a matrix of 2-forms. The relation between these two matrices is the structural equation of the connection.

The Gram–Schmidt process in linear algebra turns any C^∞ frame of a Riemannian bundle into an orthonormal frame. Relative to an orthonormal frame, the connection matrix of a metric connection is skew-symmetric. By the structural equation, the curvature matrix is also skew-symmetric. The skew-symmetry of the curvature matrix of a metric connection will have important consequences in a later chapter.

On a tangent bundle, the torsion of a connection can also be represented by a vector of 2-forms called the torsion forms. There is a structural equation relating the torsion forms to the dual forms and the connection forms.

11.1 Connection and Curvature Forms

Let ∇ be a connection on a C^∞ rank r vector bundle $\pi\colon E \to M$. We are interested in describing ∇ locally. Section 10.6 shows how on every open subset U of M, ∇ restricts to a connection on $E|_U \to U$:

$$\nabla^U : \mathfrak{X}(U) \times \Gamma(U,E) \to \Gamma(U,E).$$

We will usually omit the superscript U and write ∇^U as ∇.

Suppose U is a trivializing open set for E and $e_1, \ldots, e_r$ is a frame for E over U (Proposition 7.22), and let $X \in \mathfrak{X}(U)$ be a C^∞ vector field on U. On U, since any section $s \in \Gamma(U,E)$ is a linear combination $s = \sum a^j e_j$, the section $\nabla_X s$ can be

computed from $\nabla_X e_j$ by linearity and the Leibniz rule. As a section of E over U, $\nabla_X e_j$ is a linear combination of the e_i's with coefficients ω_j^i depending on X:

$$\nabla_X e_j = \sum \omega_j^i(X) e_i.$$

The $\mathcal{F}$-linearity of $\nabla_X e_j$ in X implies that ω_j^i is $\mathcal{F}$-linear in X and so ω_j^i is a 1-form on U (Corollary 7.27). The 1-forms ω_j^i on U are called the ***connection forms***, and the matrix $\omega = [\omega_j^i]$ is called the ***connection matrix***, of the connection ∇ relative to the frame $e_1, \ldots, e_r$ on U.

Similarly, for $X, Y \in \mathfrak{X}(U)$, the section $R(X,Y)e_j$ is a linear combination of $e_1, \ldots, e_r$:

$$R(X,Y)e_j = \sum \Omega_j^i(X,Y) e_i.$$

Since

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

is alternating and is $\mathcal{F}$-bilinear, so is Ω_j^i. By Section 7.8, Ω_j^i is a 2-form on U. The 2-forms Ω_j^i are called the ***curvature forms***, and the matrix $\Omega = [\Omega_j^i]$ is called the ***curvature matrix***, of the connection ∇ relative to the frame $e_1, \ldots, e_r$ on U.

Recall that if α and β are C^∞ 1-forms and X and Y are C^∞ vector fields on a manifold, then

$$(\alpha \wedge \beta)(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X) \tag{11.2}$$

and

$$(d\alpha)(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]) \tag{11.3}$$

([21, Section 3.7] and [21, Prop. 20.13]).

Theorem 11.1. *Let ∇ be a connection on a vector bundle $E \to M$ of rank r. Relative to a frame $e_1, \ldots, e_r$ for E over a trivializing open set U, the curvature forms Ω_j^i are related to the connection forms ω_j^i by the* ***second structural equation****:*

$$\Omega_j^i = d\omega_j^i + \sum_k \omega_k^i \wedge \omega_j^k.$$

Remark 11.2 (*The Einstein summation convention*). In classical differential geometry, it is customary to omit the summation sign $\sum$ whenever there is a pair of repeating indices, one a superscript and the other a subscript. This is called the ***Einstein summation convention***. For example, in the Einstein summation convention, $a^i e_i$ means $\sum_i a^i e_i$. We will sometimes adopt this convention if it simplifies the appearance of a proof without creating confusion.

Proof (*of Theorem 11.1*). Let X and Y be smooth vector fields on U. Then

$$\begin{aligned}
\nabla_X \nabla_Y e_j &= \nabla_X \sum_k (\omega_j^k(Y) e_k) \quad \text{(definition of connection forms)} \\
&= \sum_k X\omega_j^k(Y) e_k + \sum_k \omega_j^k(Y) \nabla_X e_k \quad \text{(Leibniz rule)} \\
&= \sum_i X\omega_j^i(Y) e_i + \sum_{i,k} \omega_j^k(Y) \omega_k^i(X) e_i.
\end{aligned}$$

Interchanging X and Y gives

$$\nabla_Y \nabla_X e_j = \sum_i Y\omega_j^i(X)e_i + \sum_{i,k} \omega_j^k(X)\omega_k^i(Y)e_i.$$

Furthermore,

$$\nabla_{[X,Y]} e_j = \sum_i \omega_j^i([X,Y])e_i.$$

Hence, in Einstein notation,

$$\begin{aligned} R(X,Y)e_j &= \nabla_X \nabla_Y e_j - \nabla_Y \nabla_X e_j - \nabla_{[X,Y]} e_j \\ &= (X\omega_j^i(Y) - Y\omega_j^i(X) - \omega_j^i([X,Y]))e_i \\ &\qquad + (\omega_k^i(X)\omega_j^k(Y) - \omega_k^i(Y)\omega_j^k(X))e_i \\ &= d\omega_j^i(X,Y)e_i + \omega_k^i \wedge \omega_j^k(X,Y)e_i \qquad \text{(by (11.3) and (11.2))} \\ &= (d\omega_j^i + \omega_k^i \wedge \omega_j^k)(X,Y)e_i. \end{aligned}$$

Comparing this with the definition of the curvature form Ω_j^i gives

$$\Omega_j^i = d\omega_j^i + \sum_k \omega_k^i \wedge \omega_j^k. \qquad \square$$

11.2 Connections on a Framed Open Set

Suppose E is a C^∞ vector bundle over a manifold M and U is an open set on which there is a C^∞ frame $e_1, \dots, e_r$ for E. We call U a ***framed open set*** for E for short. A connection ∇ on $E|_U$ determines a unique connection matrix $[\omega_j^i]$ relative to the frame $e_1, \dots, e_r$. Conversely, any matrix of 1-forms $[\omega_j^i]$ on U determines a connection on $E|_U$ as follows.

Given a matrix $[\omega_j^i]$ of 1-forms on U, and $X, Y \in \mathfrak{X}(U)$, we set

$$\nabla_X e_j = \sum \omega_j^i(X)e_i,$$

and define $\nabla_X Y$ by applying the Leibniz rule to $Y = \sum h^j e_j$:

$$\begin{aligned} \nabla_X Y = \nabla_X(h^j e_j) &= (Xh^j)e_j + h^j \omega_j^i(X)e_i \\ &= ((Xh^i) + h^j\omega_j^i(X))e_i. \end{aligned} \qquad (11.4)$$

With this definition, ∇ is a connection on $E|_U$ (Problem 11.2).

11.3 The Gram–Schmidt Process

The Gram–Schmidt process in linear algebra turns any linearly independent set of vectors $v_1, \dots, v_n$ in an inner product space V into an orthonormal set with the same span. Denote by $\operatorname{proj}_a b$ the orthogonal projection of b to the linear span of a. Then

$$\operatorname{proj}_a b = \frac{\langle b,a\rangle}{\langle a,a\rangle}a.$$

To carry out the Gram–Schmidt process, we first create an orthogonal set $w_1,\dots,w_n$:

$$\begin{aligned}
w_1 &= v_1,\\
w_2 &= v_2 - \operatorname{proj}_{w_1} v_2\\
&= v_2 - \frac{\langle v_2,w_1\rangle}{\langle w_1,w_1\rangle}w_1,\\
w_3 &= v_3 - \operatorname{proj}_{w_1} v_3 - \operatorname{proj}_{w_2} v_3\\
&= v_3 - \frac{\langle v_3,w_1\rangle}{\langle w_1,w_1\rangle}w_1 - \frac{\langle v_3,w_2\rangle}{\langle w_2,w_2\rangle}w_2,
\end{aligned} \tag{11.5}$$

and so on.

Proposition 11.3. *In the Gram–Schmidt process (11.5), for each k, the set* $w_1,\dots,w_k$ *span the same linear subspace of* V *as* $v_1,\dots,v_k$.

Proof. Let $\langle w_1,\dots,w_k\rangle$ be the linear subspace of V spanned by $w_1,\dots,w_k$. From (11.5), it is clear that each v_i is a linear combination of $w_1,\dots,w_i$. Hence, we have $\langle v_1,\dots,v_k\rangle \subset \langle w_1,\dots,w_k\rangle$.

We prove the reverse inequality by induction. The base case $\langle w_1\rangle \subset \langle v_1\rangle$ is trivially true. Suppose $\langle w_1,\dots,w_{k-1}\rangle \subset \langle v_1,\dots,v_{k-1}\rangle$. Then (11.5) shows that

$$w_k = v_k - \sum_{i=1}^{k-1}\frac{\langle v_k,w_i\rangle}{\langle w_i,w_i\rangle}w_i \in \langle v_1,\dots,v_k\rangle.$$

Hence, $\langle w_1,\dots,w_k\rangle \subset \langle v_1,\dots,v_k\rangle$. □

Since $v_1,\dots,v_k$ are linearly independent, the subspace $\langle v_1,\dots,v_k\rangle$ has dimension k. Thus, $w_1,\dots,w_k$ is a basis for $\langle v_1,\dots,v_k\rangle = \langle w_1,\dots,w_k\rangle$. In particular, none of the vectors $w_1,\dots,w_k$ is 0. It is easy to check that $w_1,\dots,w_n$ is an orthogonal set. To get an orthonormal set $e_1,\dots,e_n$, define

$$e_i = \frac{w_i}{\|w_i\|}. \tag{11.6}$$

The Gram–Schmidt process can also be applied to a frame $v_1,\dots,v_n$ for a vector bundle over an open set U. We see from (11.5) and (11.6) that it is a C^∞ process: if the sections $v_1,\dots,v_n$ of E are C^∞ over U, so are the orthonormal sections $e_1,\dots,e_n$.

11.4 Metric Connection Relative to an Orthonormal Frame

In the preceding subsections we saw that on a framed open set U for a vector bundle $E \to M$, a connection is completely specified by a matrix $[\omega^i_j]$ of 1-forms on U. Suppose now that the vector bundle $E \to M$ is endowed with a Riemannian metric. The defining property of a metric connection can be translated into a condition on the connection matrix $[\omega^i_j]$ relative to an orthonormal frame.

Proposition 11.4. *Let $E \to M$ be a Riemannian bundle and ∇ a connection on E.*

(i) *If the connection ∇ is compatible with the metric, then its connection matrix $[\omega^i_j]$ relative to any orthonormal frame $e_1, \dots, e_r$ for E over a trivializing open set $U \subset M$ is skew-symmetric.*
(ii) *If every point $p \in M$ has a trivializing neighborhood U for E such that the connection matrix $[\omega^i_j]$ relative to an orthonormal frame $e_1, \dots, e_r$ for E over U is skew-symmetric, then the connection ∇ is compatible with the metric.*

Proof. (i) Suppose ∇ is compatible with the metric. For all $X \in \mathfrak{X}(U)$ and i, j,

$$\begin{aligned} 0 = X\langle e_i, e_j\rangle &= \langle \nabla_X e_i, e_j\rangle + \langle e_i, \nabla_X e_j\rangle \\ &= \langle \omega^k_i(X) e_k, e_j\rangle + \langle e_i, \omega^k_j(X) e_k\rangle \\ &= \omega^k_i(X)\delta_{kj} + \omega^k_j(X)\delta_{ik} \qquad (\delta_{ij} = \text{Kronecker delta}) \\ &= \omega^j_i(X) + \omega^i_j(X). \end{aligned}$$

Hence,

$$\omega^j_i = -\omega^i_j.$$

(ii) We note first that compatibility with the metric is a local condition, so ∇ is compatible with the metric if and only if its restriction ∇^U to any open set U is compatible with the metric. Suppose $\omega^j_i = -\omega^i_j$. Let $s = \sum a^i e_i$ and $t = \sum b^j e_j$, with $a^i, b^j \in C^\infty(U)$. Then

$$\begin{aligned} X\langle s, t\rangle &= X\left(\sum a^i b^i\right) = \sum (Xa^i) b^i + \sum a^i X b^i. \\ \nabla_X s &= \nabla_X (a^i e_i) = (Xa^i) e_i + a^i \nabla_X e_i \\ &= (Xa^i) e_i + a^i \omega^k_i(X) e_k, \\ \langle \nabla_X s, t\rangle &= \sum (Xa^i) b^i + \sum a^i \omega^j_i(X) b^j, \qquad (11.7) \\ \langle s, \nabla_X t\rangle &= \sum (Xb^i) a^i + \sum b^i \omega^j_i(X) a^j \\ &= \sum (Xb^i) a^i + \sum a^i b^j \omega^i_j(X). \qquad (11.8) \end{aligned}$$

But by the skew-symmetry of ω,

$$\sum a^i b^j \omega^j_i(X) + \sum a^i b^j \omega^i_j(X) = 0.$$

Hence, adding (11.7) and (11.8) gives

$$\begin{aligned} \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle &= \sum (Xa^i) b^i + \sum (Xb^i) a^i \\ &= X\langle s, t\rangle. \end{aligned}$$ □

Proposition 11.5. *If the connection matrix $[\omega^i_j]$ relative to a frame $e_1, \dots, e_n$ of an affine connection on a manifold is skew-symmetric, then so is the curvature matrix $[\Omega^i_j]$.*

Proof. By the structural equation (Theorem 11.1),

$$\begin{aligned}\Omega_i^j &= d\omega_i^j + \omega_k^j \wedge \omega_i^k \\ &= -d\omega_j^i + (-\omega_j^k) \wedge (-\omega_k^i) \quad \text{(skew-symmetry of } [\omega_j^i]) \\ &= -d\omega_j^i - \omega_k^i \wedge \omega_j^k \quad \text{(anticommutativity of } \wedge) \\ &= -\Omega_j^i. \end{aligned}$$

□

By Propositions 11.4 and 11.5, if a connection on a Riemannian bundle is compatible with the metric, then its curvature matrix Ω relative to an orthonormal frame is skew-symmetric.

11.5 Connections on the Tangent Bundle

A connection on the tangent bundle TM of a manifold M is simply an affine connection on M. In addition to the curvature tensor $R(X,Y)Z$, an affine connection has a torsion tensor $T(X,Y)$.

Let U be an open set in M on which the tangent bundle TM has a smooth frame $e_1,\ldots,e_n$. If U is a coordinate open set with coordinates $x^1,\ldots,x^n$, then $\partial/\partial x^1,\ldots,\partial/\partial x^n$ is such a frame, but we will consider the more general setting where U need not be a coordinate open set. Let $\theta^1,\ldots,\theta^n$ be the dual frame of 1-forms on U; this means

$$\theta^i(e_j) = \delta^i_j.$$

Proposition 11.6. *If X is a smooth vector field on the open set U, then $X = \sum \theta^i(X)e_i$.*

Proof. Since $e_1,\ldots,e_n$ is a frame for the tangent bundle TM over U,

$$X = \sum a^j e_j$$

for some C^∞ functions $a^j \in C^\infty(U)$. Applying θ^i to both sides gives

$$\theta^i(X) = \sum_j \theta^i(a^j e_j) = \sum_j a^j \delta^i_j = a^i.$$

Therefore,

$$X = \sum \theta^i(X)e_i.$$

□

For $X,Y \in \mathfrak{X}(U)$, the torsion $T(X,Y)$ is a linear combination of the vector fields $e_1,\ldots,e_n$, so we can write

$$T(X,Y) = \sum \tau^i(X,Y)e_i.$$

Since $T(X,Y)$ is alternating and $\mathcal{F}$-bilinear, so are the coefficients τ^i. Therefore, the τ^i's are 2-forms on U, called the ***torsion forms*** of the affine connection ∇ relative to the frame $e_1,\ldots,e_n$ on U.

Since the torsion and curvature forms are determined completely by the frame $e_1,\ldots,e_n$ and the connection, there should be formulas for τ^i and Ω^i_j in terms of the dual forms and the connection forms. Indeed, Theorem 11.1 expresses the curvature forms in terms of the connection forms alone.

Theorem 11.7 (Structural equations). *Relative to a frame $e_1, \ldots, e_n$ for the tangent bundle over an open set U of a manifold M, the torsion and curvature forms of an affine connection on M can be given in terms of the dual 1-forms and the connection forms:*

(i) (the first structural equation) $\tau^i = d\theta^i + \sum_j \omega^i_j \wedge \theta^j$;
(ii) (the second structural equation) $\Omega^i_j = d\omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j$.

Proof. The second structural equation (ii) is a special case of Theorem 11.1, which is true more generally for any vector bundle. The proof of the first structural equation (i) is a matter of unraveling the definition of torsion. Let X and Y be smooth vector fields on U. By Proposition 11.6, we can write

$$Y = \sum \theta^j(Y) e_j.$$

Then

$$\begin{aligned}
\nabla_X Y &= \nabla_X(\theta^j(Y)e_j) \\
&= (X\theta^j(Y))e_j + \theta^j(Y)\nabla_X e_j && \text{(Leibniz rule)} \\
&= (X\theta^j(Y))e_j + \theta^j(Y)\omega^i_j(X)e_i && \text{(definition of connection form)}.
\end{aligned}$$

By symmetry,

$$\nabla_Y X = \big(Y\theta^j(X)\big)e_j + \theta^j(X)\omega^i_j(Y)e_i.$$

Finally, by Proposition 11.6 again,

$$[X,Y] = \theta^i([X,Y])e_i.$$

Thus,

$$\begin{aligned}
T(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y] \\
&= \Big(\big((X\theta^i(Y) - Y\theta^i(X) - \theta^i([X,Y])\big) + \big(\omega^i_j(X)\theta^j(Y) - \omega^i_j(Y)\theta^j(X)\big)\Big)e_i \\
&= (d\theta^i + \omega^i_j \wedge \theta^j)(X,Y)e_i,
\end{aligned}$$

where the last equality follows from (11.3) and (11.2). Hence,

$$\tau^i = d\theta^i + \sum \omega^i_j \wedge \theta^j. \qquad \square$$

We can now translate the two defining properties of the Riemannian connection into conditions on the matrix $[\omega^i_j]$.

Proposition 11.8. *Let M be a Riemannian manifold and U an open subset on which there is an orthonormal frame $e_1, \ldots, e_n$. Let $\theta^1, \ldots, \theta^n$ be the dual frame of 1-forms. Then there exists a unique skew-symmetric matrix $[\omega^i_j]$ of 1-forms such that*

$$d\theta^i + \sum_j \omega^i_j \wedge \theta^j = 0 \quad \textit{for all } i = 1, \ldots, n. \tag{11.9}$$

Proof. In Theorem 6.6 we showed the existence of a Riemannian connection ∇ on any manifold. Let $[\omega^i_j]$ be the connection matrix of ∇ relative to the orthonormal frame $e_1, \ldots, e_n$ on U. Because ∇ is compatible with the metric, by Proposition 11.4, the matrix $[\omega^i_j]$ is skew-symmetric. Because ∇ is torsion-free, by Theorem 11.7, it satisfies

$$d\theta^i + \sum_j \omega^i_j \wedge \theta^j = 0.$$

This proves the existence of the matrix $[\omega^i_j]$ with the two required properties.

To prove uniqueness, suppose $[\omega^i_j]$ is any skew-symmetric matrix of 1-forms on U satisfying (11.9). In Section 11.2, taking the vector bundle E to be the tangent bundle TM, we showed that $[\omega^i_j]$ defines an affine connection ∇ on U of which it is the connection matrix relative to the frame $e_1, \ldots, e_n$. Because $[\omega^i_j]$ is skew-symmetric, ∇ is compatible with the metric (Proposition 11.4), and because $[\omega^i_j]$ satisfies the equations (11.9), ∇ is torsion-free (Theorem 11.7). Thus, ∇ is the unique Riemannian connection on U. □

If $A = [\alpha^i_j]$ and $B = [\beta^i_j]$ are matrices of differential forms on M with the number of columns of A equal to the number of rows of B, then their wedge product $A \wedge B$ is defined to be the matrix of differential forms whose (i, j)-entry is

$$(A \wedge B)^i_j = \sum_k \alpha^i_k \wedge \beta^k_j,$$

and dA is defined to be $[d\alpha^i_j]$. In matrix notation, we write

$$\tau = \begin{bmatrix} \tau^1 \\ \vdots \\ \tau^n \end{bmatrix}, \quad \theta = \begin{bmatrix} \theta^1 \\ \vdots \\ \theta^n \end{bmatrix}, \quad \omega = [\omega^i_j], \quad \Omega = [\Omega^i_j].$$

Then the first structural equation becomes

$$\tau = d\theta + \omega \wedge \theta$$

and the second structural equation becomes

$$\Omega = d\omega + \omega \wedge \omega.$$

Problems

11.1. Connection and curvature forms on the Poincaré disk

The ***Poincaré disk*** is the open unit disk

$$\mathbb{D} = \{z = x + iy \in \mathbb{C} \mid |z| < 1\}$$

in the complex plane with Riemannian metric

$$\langle\ ,\ \rangle_z = \frac{4(dx\otimes dx + dy\otimes dy)}{(1-|z|^2)^2} = \frac{4(dx\otimes dx + dy\otimes dy)}{(1-x^2-y^2)^2}.$$

An orthonormal frame for $\mathbb{D}$ is

$$e_1 = \frac{1}{2}(1-|z|^2)\frac{\partial}{\partial x}, \quad e_2 = \frac{1}{2}(1-|z|^2)\frac{\partial}{\partial y}.$$

Find the connection matrix $\omega = [\omega^i_j]$ and the curvature matrix $\Omega = [\Omega^i_j]$ relative to the orthonormal frame e_1, e_2 of the Riemannian connection ∇ on the Poincaré disk. (*Hint*: First find the dual frame θ^1, θ^2. Then solve for ω^i_j in (11.9).)

11.2. Connection defined by a matrix of 1-forms
Show that (11.4) defines a connection on a vector bundle E over a framed open set U.

§12 The Theorema Egregium Using Forms

In Section 8 we proved Gauss's Theorema Egregium using vector fields. In this section we reprove the theorem, but using differential forms. An essential step is the derivation of a differential-form analogue of the Gauss curvature equation. The Theorema Egregium gives an intrinsic characterization of the Gaussian curvature of a surface, dependent only on the metric and independent of the embedding of the surface in a Euclidean space. This characterization can be taken as the definition of the Gaussian curvature of an abstract Riemannian 2-manifold. As an example, we compute the Gaussian curvature of the Poincaré half-plane.

12.1 The Gauss Curvature Equation

The Gauss curvature equation (Theorem 8.1) for an oriented surface M in $\mathbb{R}^3$ relates the curvature tensor to the shape operator. It has an analogue in terms of differential forms.

Consider a smooth surface M in $\mathbb{R}^3$ and a point p in M. Let U be an open neighborhood of p in M on which there is an orthonormal frame e_1, e_2. This is always possible by the Gram–Schmidt process, which turns any frame into an orthonormal frame. Let e_3 be the cross product $e_1 \times e_2$. Then e_1, e_2, e_3 is an orthonormal frame for the vector bundle $T\mathbb{R}^3|_U$ over U.

For the connection D on the bundle $T\mathbb{R}^3|_M$, let $[\omega^i_j]$ be the connection matrix of 1-forms relative to the orthonormal frame e_1, e_2, e_3 over U. Since D is compatible with the metric and the frame e_1, e_2, e_3 is orthonormal, the matrix $[\omega^i_j]$ is skew-symmetric (Proposition 11.4). Hence, for $X \in \mathfrak{X}(M)$,

$$\begin{aligned} D_X e_1 &= \qquad\qquad - \omega^1_2(X) e_2 - \omega^1_3(X) e_3, && (12.1)\\ D_X e_2 &= \omega^1_2(X) e_1 \qquad\qquad - \omega^2_3(X) e_3, && (12.2)\\ D_X e_3 &= \omega^1_3(X) e_1 + \omega^2_3(X) e_2. && (12.3) \end{aligned}$$

Let ∇ be the Riemannian connection on the surface M. Recall that for $X, Y \in \mathfrak{X}(M)$, the directional derivative $D_X Y$ need not be tangent to the surface M, and $\nabla_X Y$ is simply the tangential component $(D_X Y)_{\tan}$ of $D_X Y$. By (12.1) and (12.2),

$$\begin{aligned} \nabla_X e_1 &= (D_X e_1)_{\tan} = -\omega^1_2(X) e_2,\\ \nabla_X e_2 &= (D_X e_2)_{\tan} = \quad \omega^1_2(X) e_1. \end{aligned}$$

It follows that the connection matrix of the Riemannian connection ∇ on M is

$$\omega = \begin{bmatrix} 0 & \omega^1_2 \\ -\omega^1_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \omega^1_2.$$

Since

$$\omega \wedge \omega = \begin{bmatrix} 0 & \omega_2^1 \\ -\omega_2^1 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & \omega_2^1 \\ -\omega_2^1 & 0 \end{bmatrix} = \begin{bmatrix} -\omega_2^1 \wedge \omega_2^1 & 0 \\ 0 & -\omega_2^1 \wedge \omega_2^1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

the curvature matrix of ∇ is

$$\Omega = d\omega + \omega \wedge \omega = d\omega = \begin{bmatrix} 0 & d\omega_2^1 \\ -d\omega_2^1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} d\omega_2^1.$$

So the curvature matrix of ∇ is completely described by

$$\Omega_2^1 = d\omega_2^1. \tag{12.4}$$

Set the unit normal vector field N on U to be $N = -e_3$. By (12.3), the shape operator L is described by

$$L(X) = -D_X N = D_X e_3 = \omega_3^1(X)e_1 + \omega_3^2(X)e_2, \quad X \in \mathfrak{X}(M). \tag{12.5}$$

Theorem 12.1 (Gauss curvature equation). *Let e_1, e_2 be an orthonormal frame of vector fields on an oriented open subset U of a surface M in $\mathbb{R}^3$, and let e_3 be a unit normal vector field on U. Relative to e_1, e_2, e_3, the curvature form Ω_2^1 of the Riemannian connection on M is related to the connection forms of the directional derivative D on the bundle $T\mathbb{R}^3|_M$ by*

$$\Omega_2^1 = \omega_3^1 \wedge \omega_3^2. \tag{12.6}$$

We call formula (12.6) the Gauss curvature equation, because on the left-hand side, Ω_2^1 describes the curvature tensor of the surface, while on the right-hand side, ω_3^1 and ω_3^2 describe the shape operator.

Proof. Let $\tilde{\Omega}_j^i$ be the curvature forms of the connection D on $T\mathbb{R}^3|_M$. Because the curvature tensor of D is zero, the second structural equation for $\tilde{\Omega}$ gives

$$\tilde{\Omega}_j^i = d\omega_j^i + \sum_k \omega_k^i \wedge \omega_j^k = 0. \tag{12.7}$$

In particular,

$$d\omega_2^1 + \omega_1^1 \wedge \omega_2^1 + \omega_2^1 \wedge \omega_2^2 + \omega_3^1 \wedge \omega_2^3 = 0.$$

Since $\omega_1^1 = \omega_2^2 = 0$, this reduces to

$$d\omega_2^1 + \omega_3^1 \wedge \omega_2^3 = 0.$$

Since the matrix $[\omega_j^i]$ is skew-symmetric,

$$d\omega_2^1 = \omega_3^1 \wedge \omega_3^2.$$

The Gauss curvature equation now follows from (12.4).

12.2 The Theorema Egregium

Keeping the same notations as in the preceding subsection, we will derive formulas for the Gaussian curvature of a surface in $\mathbb{R}^3$, first in terms of the connection forms for the directional derivative and then in terms of the curvature form Ω_2^1.

Proposition 12.2. *For a smooth surface in $\mathbb{R}^3$, if e_1, e_2 is an orthonormal frame over an oriented open subset U of the surface and e_3 is a unit normal vector field on U, then the Gaussian curvature K on U is given by*

$$K = \det \begin{bmatrix} \omega_3^1(e_1) & \omega_3^1(e_2) \\ \omega_3^2(e_1) & \omega_3^2(e_2) \end{bmatrix} = (\omega_3^1 \wedge \omega_3^2)(e_1, e_2).$$

Proof. From (12.5),

$$L(e_1) = \omega_3^1(e_1)e_1 + \omega_3^2(e_1)e_2,$$
$$L(e_2) = \omega_3^1(e_2)e_1 + \omega_3^2(e_2)e_2.$$

So the matrix of L relative to the frame e_1, e_2 is

$$\begin{bmatrix} \omega_3^1(e_1) & \omega_3^1(e_2) \\ \omega_3^2(e_1) & \omega_3^2(e_2) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} K = \det L &= \det \begin{bmatrix} \omega_3^1(e_1) & \omega_3^1(e_2) \\ \omega_3^2(e_1) & \omega_3^2(e_2) \end{bmatrix} \\ &= \omega_3^1(e_1)\omega_3^2(e_2) - \omega_3^1(e_2)\omega_3^2(e_1) = (\omega_3^1 \wedge \omega_3^2)(e_1, e_2). \end{aligned}$$ □

Theorem 12.3 (Theorema Egregium). *For a smooth surface in $\mathbb{R}^3$, if e_1, e_2 is an orthonormal frame over an open subset U of the surface with dual frame θ^1, θ^2, then the Gaussian curvature K on U is given by*

$$K = \Omega_2^1(e_1, e_2) \tag{12.8}$$

or by

$$d\omega_2^1 = K\theta^1 \wedge \theta^2. \tag{12.9}$$

Proof. Formula (12.8) is an immediate consequence of Proposition 12.2 and the Gauss curvature equation (12.6).

As for (12.9), since

$$K = K(\theta^1 \wedge \theta^2)(e_1, e_2) = \Omega_2^1(e_1, e_2)$$

and a 2-form on U is completely determined by its value on e_1, e_2, we have $\Omega_2^1 = K\theta^1 \wedge \theta^2$. By (12.4),

$$d\omega_2^1 = K\theta^1 \wedge \theta^2.$$ □

Since Ω_2^1 depends only on the metric and not on the embedding of the surface in $\mathbb{R}^3$, formula (12.8) shows that the same is true of the Gaussian curvature. We can therefore take this formula to be the definition of the Gaussian curvature of an abstract Riemannian 2-manifold. It is consistent with the formula in the first version (Theorem 8.3) of the Theorema Egregium, for by the definition of the curvature matrix, if e_1, e_2 is an orthonormal frame on an open subset U of M, then

$$R(X,Y)e_2 = \Omega_2^1(X,Y)e_1 \quad \text{for all } X,Y \in \mathfrak{X}(U).$$

so that

$$\langle R(e_1,e_2)e_2, e_1\rangle = \langle \Omega_2^1(e_1,e_2)e_1, e_1\rangle = \Omega_2^1(e_1,e_2).$$

Definition 12.4. The ***Gaussian curvature*** K at a point p of a Riemannian 2-manifold M is defined to be

$$K_p = \langle R_p(u,v)v, u\rangle \tag{12.10}$$

for any orthonormal basis u, v for the tangent plane T_pM.

For the Gaussian curvature to be well defined, we need to show that formula (12.10) is independent of the choice of orthonormal basis. This we do in the next section.

12.3 Skew-Symmetries of the Curvature Tensor

Recall that the curvature of an affine connection ∇ on a manifold M is defined to be

$$R\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M),$$
$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

We showed that $R(X,Y)Z$ is $\mathcal{F}$-linear in every argument; therefore, it is a point operator. A point operator is also called a ***tensor***. It is immediate from the definition that the curvature tensor $R(X,Y)Z$ is skew-symmetric in X and Y.

Proposition 12.5. *If an affine connection ∇ on a Riemannian manifold M is compatible with the metric, then for vector fields X, Y, Z, $W \in \mathfrak{X}(M)$, the tensor $\langle R(X,Y)Z, W\rangle$ is skew-symmetric in Z and W.*

Proof. Because $\langle R(X,Y)Z, W\rangle$ is a tensor, it is enough to check its skew-symmetry locally, for example, on an open set U on which there is a frame $e_1,\ldots,e_n$. By Gram–Schmidt, we may assume that the frame $e_1,\ldots,e_n$ is orthonormal. Then

$$\langle R(X,Y)e_j, e_i\rangle = \Omega_j^i(X,Y).$$

Since ∇ is compatible with the metric, its curvature matrix $\Omega = [\Omega_j^i]$ relative to an orthonormal frame is skew-symmetric. Therefore,

$$\langle R(X,Y)e_j, e_i\rangle = \Omega_j^i(X,Y) = -\Omega_i^j(X,Y) = -\langle R(X,Y)e_i, e_j\rangle.$$

On U, we can write $Z=\sum z^i e_i$ and $W=\sum w^j e_j$ for some C^∞ functions z^i, $w^j \in C^\infty(U)$. Then

$$\begin{aligned}\langle R(X,Y)Z,W\rangle &= \sum z^i w^j \langle R(X,Y)e_i, e_j\rangle = -\sum z^i w^j \langle R(X,Y)e_j, e_i\rangle \\ &= -\langle R(X,Y)W,Z\rangle.\end{aligned}$$ □

We now show that $\langle R(u,v)v,u\rangle$ is independent of the orthonormal basis u,v for T_pM. Suppose $\bar{u},\bar{v}$ is another orthonormal basis. Then

$$\begin{aligned}\bar{u} &= au+bv,\\ \bar{v} &= cu+dv\end{aligned}$$

for an orthogonal matrix $A=\begin{bmatrix} a & b \\ c & d\end{bmatrix}$, and

$$\begin{aligned}\langle R(\bar{u},\bar{v})\bar{v},\bar{u}\rangle &= \langle (\det A)R(u,v)(cu+dv), au+dv\rangle \\ &= (\det A)^2 \langle R(u,v)v,u\rangle\end{aligned}$$

by the skew-symmetry of $\langle R(u,v)z,w\rangle$ in z and w. Since $A\in \mathrm{O}(2)$, $\det A=\pm 1$. Hence,

$$\langle R(\bar{u},\bar{v})\bar{v},\bar{u}\rangle = \langle R(u,v)v,u\rangle.$$

12.4 Sectional Curvature

Let M be a Riemannian manifold and p a point in M. If P is a 2-dimensional subspace of the tangent space T_pM, then we define the ***sectional curvature*** of P to be

$$K(P) = \langle R(e_1,e_2)e_2,e_1\rangle \tag{12.11}$$

for any orthonormal basis e_1,e_2 of P. Just as in the definition of the Gaussian curvature, the right-hand side of (12.11) is independent of the orthonormal basis e_1,e_2 (cf§12.3).

If u,v is an arbitrary basis for the 2-plane P, then a computation similar to that in §8.3 shows that the sectional curvature of P is also given by

$$K(P) = \frac{\langle R(u,v)v,u\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}.$$

12.5 Poincaré Half-Plane

Example 12.6 (*The Gaussian curvature of the Poincaré half-plane*). The Poincaré half-plane is the upper half-plane

$$\mathbb{H}^2 = \{(x,y)\in \mathbb{R}^2 \mid y>0\}$$

with the metric:

$$\langle\ ,\ \rangle_{(x,y)} = \frac{dx\otimes dx + dy\otimes dy}{y^2}.$$

Classically, the notation for a Riemannian metric is ds^2. Hence, the metric on the Poincaré half-plane is

$$ds^2 = \frac{dx\otimes dx + dy\otimes dy}{y^2}.$$

With this metric, an orthonormal frame is

$$e_1 = y\frac{\partial}{\partial x}, \quad e_2 = y\frac{\partial}{\partial y}.$$

So the dual frame is

$$\theta^1 = \frac{1}{y}\,dx, \quad \theta^2 = \frac{1}{y}\,dy.$$

Hence,

$$d\theta^1 = \frac{1}{y^2}\,dx\wedge dy, \quad d\theta^2 = 0. \tag{12.12}$$

On the Poincaré half-plane the connection form ω_2^1 is a linear combination of dx and dy, so we may write

$$\omega_2^1 = a\,dx + b\,dy. \tag{12.13}$$

We will determine the coefficients a and b from the first structural equation:

$$d\theta^1 = -\omega_2^1\wedge\theta^2, \tag{12.14}$$

$$d\theta^2 = -\omega_1^2\wedge\theta^1 = \omega_2^1\wedge\theta^1. \tag{12.15}$$

By (12.12), (12.13), and (12.14),

$$\frac{1}{y^2}\,dx\wedge dy = d\theta^1 = -(a\,dx + b\,dy)\wedge\frac{1}{y}\,dy = -\frac{a}{y}\,dx\wedge dy.$$

So $a = -1/y$. By (12.12), (12.13), and (12.15),

$$0 = d\theta^2 = (-\frac{1}{y}\,dx + b\,dy)\wedge\frac{1}{y}\,dx = -\frac{b}{y}\,dx\wedge dy.$$

So $b = 0$. Therefore,

$$\begin{aligned}\omega_2^1 &= -\frac{1}{y}\,dx,\\ d\omega_2^1 &= \frac{1}{y^2}\,dy\wedge dx\\ &= -\frac{1}{y^2}\,dx\wedge dy.\end{aligned}$$

By definition, the Gaussian curvature of the Poincaré half-plane is

$$\begin{aligned} K = \Omega^1_2(e_1, e_2) &= -\frac{1}{y^2}(dx \wedge dy)\left(y\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\right) \\ &= -(dx \wedge dy)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = -1. \end{aligned}$$

Problems

12.1. The orthogonal group $\mathrm{O}(2)$

(a) Show that an element A of the orthogonal group $\mathrm{O}(2)$ is either

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{ or } \begin{bmatrix} a & b \\ b & -a \end{bmatrix},$$

where $a^2 + b^2 = 1$, $a, b \in \mathbb{R}$.

(b) Let $\mathrm{SO}(2) = \{A \in \mathrm{O}(2) \mid \det A = 1\}$. Show that every element A of $\mathrm{SO}(2)$ is of the form

$$A = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \quad t \in \mathbb{R}.$$

Thus, $\mathrm{SO}(2)$ is the group of rotations about the origin in $\mathbb{R}^2$.

Let $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $\mathrm{O}(2) = \mathrm{SO}(2) \cup \mathrm{SO}(2)J$. This proves that $\mathrm{O}(2)$ has two connected components, each diffeomorphic to the circle S^1.

12.2. Gaussian curvature of the Euclidean plane
Let $e_1 = \partial/\partial x, e_2 = \partial/\partial y$ be the standard orthonormal frame on $\mathbb{R}^2$. Compute the connection form ω^1_2 relative to e_1, e_2 of the Riemannian connection on $\mathbb{R}^2$, and then compute the Gaussian curvature C.

12.3. Gaussian curvature of a non-Euclidean plane
Let $h(x)$ be a C^∞ positive function on $\mathbb{R}$; thus, $h(x) > 0$ for all $x \in \mathbb{R}$. At each $(x, y) \in \mathbb{R}^2$, define

$$\langle\ ,\ \rangle_{(x,y)} = dx \otimes dx + h(x)^2\, dy \otimes dy.$$

(a) Show that $\langle\ ,\ \rangle$ is a Riemannian metric on $\mathbb{R}^2$.

(b) Compute the Gaussian curvature of $\mathbb{R}^2$ with this metric.

12.4. Gaussian curvature of the Poincaré disk
Compute the Gaussian curvature of the Poincaré disk defined in Problem 11.1.

12.5. Gaussian curvature under a conformal map
Let $T: M \to M'$ be a diffeomorphism of Riemannian manifolds of dimension 2. Suppose at each point $p \in M$, there is a positive number $a(p)$ such that

$$\langle T_* v, T_* w\rangle_{M', T(p)} = a(p)\langle v, w\rangle_{M,p}$$

for all $u, v \in T_p(M)$. Find the relation between the Gaussian curvatures of M and M'.

Chapter 3

Geodesics

A geodesic on a Riemannian manifold is the analogue of a line in Euclidean space. One can characterize a line in Euclidean space in several equivalent ways, among which are the following:

(1) A line is "straight" in the sense that it has a parametrization with a constant velocity vector field.
(2) A line connecting two points gives the shortest distance between the two points.

These two properties are not necessarily equivalent on a Riemannian manifold. We define a geodesic by generalizing the notion of "straightness." For this, it is not necessary to have a metric, but only a connection. On a Riemannian manifold, of course, there is always the unique Riemannian connection, and so one can speak of geodesics on a Riemannian manifold.

§13 More on Affine Connections

This chapter is a compilation of some properties of an affine connection that will prove useful later. First we discuss how an affine connection on a manifold M induces a unique ***covariant derivative*** of vector fields along a smooth curve in M. This generalizes the derivative dV/dt of a vector field V along a smooth curve in $\mathbb{R}^n$. Secondly, we present a way of describing a connection in local coordinates, using the so-called ***Christoffel symbols***.

13.1 Covariant Differentiation Along a Curve

Let $c\colon [a,b] \to M$ be a smooth parametrized curve in a manifold M. Recall that a vector field along the curve c in M is a function

$$V\colon [a,b] \to \coprod_{t\in[a,b]} T_{c(t)}M,$$

L.W. Tu, *Differential Geometry*, Graduate Texts in Mathematics 275,
DOI 10.1007/978-3-319-55084-8_3

where $\coprod$ stands for the disjoint union, such that $V(t) \in T_{c(t)}M$ (Figure 13.1). Such a vector field $V(t)$ is C^∞ if for any C^∞ function f on M, the function $V(t)f$ is C^∞ as a function of t. We denote the vector space of all C^∞ vector fields along the curve $c(t)$ by $\Gamma(TM|_{c(t)})$.

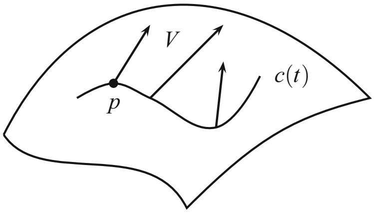


Fig. 13.1. Vector field along a curve $c(t)$ in M.

For a smooth vector field $V(t) = \sum v^i(t)\partial/\partial x^i$ along a smooth curve $c(t)$ in $\mathbb{R}^n$, we defined its derivative dV/dt by differentiating the components $v^i(t)$ with respect to t. Then $dV/dt = \sum \dot{v}^i \partial/\partial x^i$ satisfies the following properties:

(i) dV/dt is $\mathbb{R}$-linear in V;
(ii) for any C^∞ function f on $[a,b]$,

$$\frac{d(fV)}{dt} = \frac{df}{dt}V + f\frac{dV}{dt};$$

(iii) (Proposition 4.11) if V is induced from a C^∞ vector field $\tilde{V}$ on $\mathbb{R}^n$, in the sense that $V(t) = \tilde{V}_{c(t)}$ and D is the directional derivative in $\mathbb{R}^n$, then

$$\frac{dV}{dt} = D_{c'(t)}\tilde{V}.$$

It turns out that to every connection ∇ on a manifold M one can associate a way of differentiating vector fields along a curve satisfying the same properties as the derivative above.

Theorem 13.1. *Let M be a manifold with an affine connection ∇, and $c\colon [a,b] \to M$ a smooth curve in M. Then there is a unique map*

$$\frac{D}{dt}\colon \Gamma(TM|_{c(t)}) \to \Gamma(TM|_{c(t)})$$

such that for $V \in \Gamma(TM|_{c(t)})$,

(i) ($\mathbb{R}$-linearity) *DV/dt is $\mathbb{R}$-linear in V;*
(ii) (Leibniz rule) *for any C^∞ function f on the interval $[a,b]$,*

$$\frac{D(fV)}{dt} = \frac{df}{dt}V + f\frac{DV}{dt};$$

(iii) (Compatibility with ∇) *if V is induced from a C^∞ vector field $\tilde{V}$ on M, in the sense that $V(t) = \tilde{V}_{c(t)}$, then*

$$\frac{DV}{dt}(t) = \nabla_{c'(t)}\tilde{V}.$$

We call DV/dt the ***covariant derivative*** (associated to ∇) of the vector field V along the curve $c(t)$ in M.

Proof. Suppose such a covariant derivative D/dt exists. On a framed open set $(U, e_1, \ldots, e_n)$, a vector field $V(t)$ along c can be written as a linear combination

$$V(t) = \sum v^i(t) e_{i,c(t)}.$$

Then

$$\begin{aligned} \frac{DV}{dt} &= \sum \frac{D}{dt}\left(v^i(t) e_{i,c(t)}\right) && \text{(by property (i))} \\ &= \sum \frac{dv^i}{dt} e_i + v^i \frac{De_i}{dt} && \text{(by property (ii))} \\ &= \sum \frac{dv^i}{dt} e_i + v^i \nabla_{c'(t)} e_i && \text{(by property (iii))}, \end{aligned} \tag{13.1}$$

where we abuse notation and write e_i instead of $e_{i,c(t)}$. This formula proves the uniqueness of D/dt if it exists.

As for existence, we define DV/dt for a curve $c(t)$ in a framed open set U by the formula (13.1). It is easily verified that DV/dt satisfies the three properties (i), (ii), (iii) (Problem 13.1). Hence, D/dt exists for curves in U. If $\bar{e}_1, \ldots, \bar{e}_n$ is another frame on U, then $V(t)$ is a linear combination $\sum \bar{v}^i(t)\bar{e}_{i,c(t)}$ and the covariant derivative $\bar{D}V/dt$ defined by

$$\frac{\bar{D}V}{dt} = \sum_i \frac{d\bar{v}^i}{dt}\bar{e}_i + \bar{v}^i \nabla_{c'(t)}\bar{e}_i$$

also satisfies the three properties of the theorem. By the uniqueness of the covariant derivative, $DV/dt = \bar{D}V/dt$. This proves that the covariant derivative DV/dt is independent of the frame. By covering M with framed open sets, (13.1) then defines a covariant derivative DV/dt for the curve $c(t)$ in M. □

Theorem 13.2. *Let M be a Riemannian manifold, ∇ an affine connection on M, and $c\colon [a,b] \to M$ a smooth curve in M. If ∇ is compatible with the metric, then for any smooth vector fields V, W along c,*

$$\frac{d}{dt}\langle V, W\rangle = \left\langle \frac{DV}{dt}, W \right\rangle + \left\langle V, \frac{DW}{dt} \right\rangle.$$

Proof. It suffices to check this equality locally, so let U be an open set on which an orthonormal frame $e_1, \ldots, e_n$ exists. With respect to this frame,

$$V = \sum v^i(t) e_{i,c(t)}, \quad W = \sum w^j(t) e_{j,c(t)}$$

for some C^∞ functions v^i, w^j on $[a,b]$. Then

$$\frac{d}{dt}\langle V,W\rangle = \frac{d}{dt}\sum v^i w^i = \sum \frac{dv^i}{dt} w^i + \sum v^i \frac{dw^i}{dt}.$$

By the defining properties of a covariant derivative,

$$\frac{DV}{dt} = \sum_i \frac{dv^i}{dt} e_i + v^i \frac{De_i}{dt} = \sum_i \frac{dv^i}{dt} e_i + v^i \nabla_{c'(t)} e_i,$$

where we again abuse notation and write e_i instead of $e_i \circ c$. Similarly,

$$\frac{DW}{dt} = \frac{dw^j}{dt} e_j + w^j \nabla_{c'(t)} e_j.$$

Hence,

$$\left\langle \frac{DV}{dt}, W \right\rangle + \left\langle V, \frac{DW}{dt} \right\rangle = \sum_i \frac{dv^i}{dt} w^i + \sum_{i,j} v^i w^j \langle \nabla_{c'(t)} e_i, e_j \rangle + \sum_i v^i \frac{dw^i}{dt} + \sum_{i,j} v^i w^j \langle e_i, \nabla_{c'(t)} e_j \rangle.$$

Since $e_1, \ldots, e_n$ are orthonormal vector fields on U and ∇ is compatible with the metric,

$$\langle \nabla_{c'(t)} e_i, e_j \rangle + \langle e_i, \nabla_{c'(t)} e_j \rangle = c'(t)\langle e_i, e_j \rangle = c'(t)\delta_{ij} = 0.$$

Therefore,

$$\left\langle \frac{DV}{dt}, W \right\rangle + \left\langle V, \frac{DW}{dt} \right\rangle = \sum \frac{dv^i}{dt} w^i + v^i \frac{dw^i}{dt} = \frac{d}{dt}\langle V,W\rangle.$$ □

Example 13.3. If ∇ is the directional derivative on $\mathbb{R}^n$ and $V(t) = \sum v^i(t)\partial/\partial x^i$ is a vector field along a smooth curve $c(t)$ in $\mathbb{R}^n$, then the covariant derivative is

$$\frac{DV}{dt} = \sum \frac{dv^i}{dt}\frac{\partial}{\partial x^i} + \sum v^i D_{c'(t)} \frac{\partial}{\partial x^i} = \sum \frac{dv^i}{dt}\frac{\partial}{\partial x^i} = \frac{dV}{dt},$$

since $D_{c'(t)}\partial/\partial x^i = 0$ by (4.2).

13.2 Connection-Preserving Diffeomorphisms

Although it is in general not possible to push forward a vector field except under a diffeomorphism, it is always possible to push forward a vector field along a curve under any C^∞ map. Let $f\colon M \to \tilde{M}$ be a C^∞ map (not necessarily a diffeomorphism) of manifolds, and $c\colon [a,b] \to M$ a smooth curve in M. The ***pushforward*** of the vector field $V(t)$ along c in M is the vector field $(f_*V)(t)$ along the image curve $f \circ c$ in $\tilde{M}$ defined by

$$(f_*V)(t) = f_{*,c(t)}\big(V(t)\big).$$

Denote by (M,∇) a C^∞ manifold with an affine connection ∇. We say that a C^∞ diffeomorphism $f\colon (M,\nabla)\to(\tilde{M},\tilde{\nabla})$ ***preserves the connection*** or ***is connection-preserving*** if for all $X,Y\in\mathfrak{X}(M)$,

$$f_*(\nabla_X Y)=\tilde{\nabla}_{f_*X}f_*Y.$$

In this terminology, an isometry of Riemannian manifolds preserves the Riemannian connection (Problem 8.2).

We now show that a connection-preserving diffeomorphism also preserves the covariant derivative along a curve.

Proposition 13.4. *Suppose $f\colon (M,\nabla)\to(\tilde{M},\tilde{\nabla})$ is a connection-preserving diffeomorphism, $c(t)$ a smooth curve in M, and D/dt, $\tilde{D}/dt$ the covariant derivatives along c in M and $f\circ c$ in $\tilde{M}$, respectively. If $V(t)$ is a vector field along c in M, then*

$$f_*\left(\frac{DV}{dt}\right)=\frac{\tilde{D}(f_*V)}{dt}.$$

Proof. Choose a neighborhood U of $c(t)$ on which there is a frame $e_1,\dots,e_n$, and write $V(t)=\sum v^i(t)e_{i,c(t)}$. Then $\tilde{e}_1:=f_*e_1,\dots,\tilde{e}_n:=f_*e_n$ is a frame on the neighborhood $f(U)$ of $f(c(t))$ in $\tilde{M}$ and

$$(f_*V)(t)=\sum v^i(t)\tilde{e}_{i,(f\circ c)(t)}.$$

By the definition of the covariant derivative,

$$\frac{DV}{dt}(t)=\sum\frac{dv^i}{dt}(t)e_{i,c(t)}+v^i(t)\nabla_{c'(t)}e_i.$$

Because f preserves the connection,

$$\begin{aligned}\left(f_*\frac{DV}{dt}\right)(t)&=\sum\frac{dv^i}{dt}f_{*,c(t)}(e_{i,c(t)})+v^i(t)\tilde{\nabla}_{f_*c'(t)}f_*e_i\\&=\sum\frac{dv^i}{dt}\tilde{e}_{i,(f\circ c)(t)}+v^i(t)\tilde{\nabla}_{(f\circ c)'(t)}\tilde{e}_i\\&=\left(\frac{\tilde{D}}{dt}f_*V\right)(t).\end{aligned}$$

□

13.3 Christoffel Symbols

One way to describe a connection locally is by the connection forms relative to a frame. Another way, which we now discuss, is by a set of n^3 functions called the ***Christoffel symbols***.

The Christoffel symbols are defined relative to a coordinate frame. Let ∇ be an affine connection on a manifold M and let $(U, x^1, \ldots, x^n)$ be a coordinate open set in M. Denote by ∂_i the coordinate vector field $\partial/\partial x^i$. Then $\nabla_{\partial_i}\partial_j$ is a linear combination of $\partial_1, \ldots, \partial_n$, so there exist numbers Γ_{ij}^k at each point such that

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^{n} \Gamma_{ij}^k \partial_k.$$

These n^3 functions Γ_{ij}^k are called the ***Christoffel symbols*** of the connection ∇ on the coordinate open set $(U, x^1, \ldots, x^n)$. By the Leibniz rule and $\mathcal{F}$-linearity in the first argument of a connection, the Christoffel symbols completely describe a connection on U.



Elwin Bruno Christoffel

(1829–1900)

Proposition 13.5. *An affine connection ∇ on a manifold is torsion-free if and only if in every coordinate chart $(U, x^1, \ldots, x^n)$ the Christoffel symbol Γ_{ij}^k is symmetric in i and j:*

$$\Gamma_{ij}^k = \Gamma_{ji}^k.$$

Proof. $(\Rightarrow)$ Let $(U, x^1, \ldots, x^n)$ be a coordinate open set. Since partial differentiation is independent of the order of differentiation,

$$[\partial_i, \partial_j] = \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0.$$

By torsion-freeness,

$$\nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i = [\partial_i, \partial_j] = 0.$$

In terms of Christoffel symbols,

$$\sum_k \Gamma_{ij}^k \partial_k - \sum_k \Gamma_{ji}^k \partial_k = 0.$$

Since $\partial_1, \ldots, \partial_n$ are linearly independent at each point, $\Gamma_{ij}^k = \Gamma_{ji}^k$.

$(\Leftarrow)$ Conversely, suppose $\Gamma_{ij}^k = \Gamma_{ji}^k$ in the coordinate chart $(U, x^1, \ldots, x^n)$. Then $\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i$. Hence,

$$T(\partial_i, \partial_j) = \nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i = 0.$$

Since $T(\ ,\ )$ is a bilinear function on T_pM, this proves that $T(X,Y)_p := T(X_p, Y_p) = 0$ for all $X_p, Y_p \in T_pM$. Thus, for all $X, Y \in \mathfrak{X}(M)$, we have $T(X,Y) = 0$. □

Remark 13.6. Because of this proposition, a torsion-free connection is also called a ***symmetric*** connection.

Example 13.7 (*Christoffel symbols for the Poincaré half-plane*). The Poincaré half-plane (Example 12.6) is covered by a single coordinate open set (U, x, y). Let $\partial_1 = \partial/\partial x$, $\partial_2 = \partial/\partial y$ be the coordinate frame. We showed in Example 12.6 that its connection form ω_2^1 relative to the orthonormal frame $e_1 = y\partial_1, e_2 = y\partial_2$ is

$$\omega_2^1 = -\frac{1}{y}\,dx.$$

Then

$$\partial_1 = \frac{1}{y}e_1, \quad \partial_2 = \frac{1}{y}e_2,$$

and for any smooth vector field X on the Poincaré half-plane,

$$\begin{aligned}
\nabla_X \partial_1 = \nabla_X\left(\frac{1}{y}e_1\right) &= X\left(\frac{1}{y}\right)e_1 + \frac{1}{y}\nabla_X e_1 \\
&= -\frac{1}{y^2}(Xy)y\partial_1 + \frac{1}{y}\omega_1^2(X)e_2 \\
&= -\frac{1}{y}(Xy)\partial_1 + \frac{1}{y^2}dx(X)y\partial_2 \\
&= -\frac{1}{y}(Xy)\partial_1 + \frac{1}{y}(Xx)\partial_2 \qquad (13.2)
\end{aligned}$$

Similarly,

$$\nabla_X \partial_2 = -\frac{1}{y}(Xx)\partial_1 - \frac{1}{y}(Xy)\partial_2. \qquad (13.3)$$

By (13.2) and (13.3), relative to the coordinate frame ∂_1, ∂_2, the connection is given by

$$\begin{aligned}
\nabla_{\partial_1}\partial_1 &= \frac{1}{y}\partial_2, & \nabla_{\partial_2}\partial_1 &= -\frac{1}{y}\partial_1, \\
\nabla_{\partial_1}\partial_2 &= -\frac{1}{y}\partial_1, & \nabla_{\partial_2}\partial_2 &= -\frac{1}{y}\partial_2.
\end{aligned}$$

Therefore, the Christoffel symbols Γ_{ij}^k for the Poincaré half-plane are

ij \ k	1	2
11	0	$1/y$
12	$-1/y$	0
21	$-1/y$	0
22	0	$-1/y$

Problems

13.1. Covariant derivative along a curve
Verify that if DV/dt is defined by (13.1), then it satisfies the three properties (i), (ii), and (iii) of Theorem 13.1.

13.2. Covariant derivative on a surface in $\mathbb{R}^3$
Let M be a surface in $\mathbb{R}^3$ with its Riemannian connection: for $X, Y \in \mathfrak{X}(M)$,

$$\nabla_X Y = \mathrm{pr}(D_X Y).$$

If $V(t)$ is a vector field along a curve $c(t)$ in M, show that $DV/dt = \mathrm{pr}(dV/dt)$. (*Hint*: Show that $\mathrm{pr}(dV/dt)$ verifies the three properties of the covariant derivative.)

13.3. Chain rule for the covariant derivative
Suppose M is a manifold with a connection and DV/dt is the covariant derivative of a vector field V along a curve $c(t)$ in M. If t is a C^∞ function of another variable u, then V gives rise to a vector field $V(t(u))$ along the curve $\bar{c}(u) = c(t(u))$. We will write $\bar{V}(u) = V\big(t(u)\big)$. Show that

$$\frac{D\bar{V}}{du} = \frac{DV}{dt}\frac{dt}{du}.$$

13.4. Christoffel symbols of a surface of revolution
Let M be a surface of revolution in $\mathbb{R}^3$ with parametrization

$$\psi(u,v) = \begin{bmatrix} f(u)\cos v \\ f(u)\sin v \\ g(u) \end{bmatrix}, \quad 0 < v < 2\pi$$

as in Problem 5.7. Let ∇ be the Riemannian connection on M. Find the Christoffel symbols of ∇ with respect to u, v.

13.5. Christoffel symbols of the Poincaré disk
Compute the Christoffel symbols of the Riemannian connection on the Poincaré disk $\mathbb{D}$ in Problem 11.1.

§14 Geodesics

In this section we prove the existence and uniqueness of a geodesic with a specified initial point and a specified initial velocity. As an example, we determine all the geodesics on the Poincaré half-plane. These geodesics played a pivotal role in the history of non-Euclidean geometry, for they proved for the first time that the parallel postulate is independent of Euclid's first four postulates.

14.1 The Definition of a Geodesic

A straight line in $\mathbb{R}^n$ with parametrization

$$c(t) = p + tv, \quad p, v \in \mathbb{R}^n,$$

is characterized by the property that its acceleration $c''(t)$ is identically zero. If $T(t) = c'(t)$ is the tangent vector of the curve at $c(t)$, then $c''(t)$ is also the covariant derivative DT/dt associated to the Euclidean connection D on $\mathbb{R}^n$. This example suggests a way to generalize the notion of "straightness" to an arbitrary manifold with a connection.

Definition 14.1. Let M be a manifold with an affine connection ∇ and I an open, closed, or half-open interval in $\mathbb{R}$. A parametrized curve $c\colon I \to M$ is a ***geodesic*** if the covariant derivative DT/dt of its velocity vector field $T(t) = c'(t)$ is zero. The geodesic is said to be ***maximal*** if its domain I cannot be extended to a larger interval.

Remark 14.2. The notion of a geodesic depends only on a connection and does not require a metric on the manifold M. However, if M is a Riemannian manifold, then we will always take the connection to be the unique Riemannian connection. On a Riemannian manifold, the ***speed*** of a curve $c(t)$ is defined to be the magnitude of its velocity vector:

$$\|c'(t)\| = \sqrt{\langle c'(t), c'(t)\rangle}.$$

Proposition 14.3. *The speed of a geodesic on a Riemannian manifold is constant.*

Proof. Let $T = c'(t)$ be the velocity of the geodesic. The speed is constant if and only if its square $f(t) = \langle T, T\rangle$ is constant. But

$$\begin{aligned} f'(t) &= \frac{d}{dt}\langle T, T\rangle \\ &= 2\left\langle \frac{DT}{dt}, T\right\rangle = 0. \end{aligned}$$

So $f(t)$ is constant. □

Let M be a smooth surface in $\mathbb{R}^3$. Recall that its Riemannian connection is given by

$$\nabla_X Y = (D_X Y)_{\tan},$$

the tangential component of the directional derivative $D_X Y$. If N is a unit normal vector field on an open subset U of M, then the tangential component of a vector field Z along M is

$$Z_{\tan} = Z - \langle Z, N\rangle N.$$

By Problem 13.2, for a vector field V along a curve $c(t)$ in M, the covariant derivative associated to the Riemannian connection ∇ on M is

$$\frac{DV}{dt} = \left(\frac{dV}{dt}\right)_{\tan}.$$

Example 14.4 (*Geodesics on a sphere*). On a 2-sphere M of radius a in $\mathbb{R}^3$, let $\gamma(t)$ be a great circle parametrized by arc length. Then $\gamma(t)$ has unit speed. Differentiating

$$\langle \gamma'(t), \gamma'(t)\rangle = 1$$

with respect to t gives

$$2\langle \gamma''(t), \gamma'(t)\rangle = 0.$$

This shows that the acceleration $\gamma''(t)$ of a unit-speed curve in $\mathbb{R}^3$ is perpendicular to the velocity.

Since $\gamma(t)$ lies in the plane of the circle, so do $\gamma'(t)$ and $\gamma''(t)$. Being perpendicular to $\gamma'(t)$, the acceleration $\gamma''(t)$ must point in the radial direction (Figure 14.1). Hence, because $\gamma''(t)$ is perpendicular to the tangent plane at $\gamma(t)$,

$$\frac{DT}{dt} = \left(\frac{dT}{dt}\right)_{\tan} = \gamma''(t)_{\tan} = 0.$$

This shows that every great circle is a geodesic on the sphere.

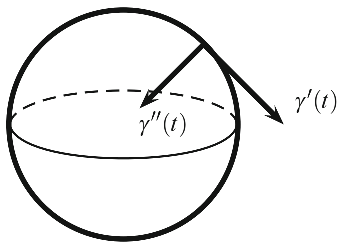


Fig. 14.1. Velocity and acceleration vectors of a great circle.

14.2 Reparametrization of a Geodesic

In this section we show that a geodesic can be reparametrized as long as the reparametrization is a linear function.

Proposition 14.5. *Suppose $\gamma(u)$ is a nonconstant geodesic on a manifold with a connection and $\bar{\gamma}(t) := \gamma(u(t))$ is a reparametrization of $\gamma(u)$. Then $\bar{\gamma}(t)$ is a geodesic if and only if $u = \alpha t + \beta$ for some real constants α and β.*

Proof. Let $T(u) = \gamma'(u)$ and $\bar{T}(t) = \bar{\gamma}'(t)$ be the tangent vector fields of the two curves $\gamma(u)$ and $\bar{\gamma}(t)$. By the chain rule,

$$\begin{aligned}\bar{T}(t) &= \frac{d}{dt}\gamma(u(t)) = \gamma'(u(t))u'(t)\\ &= u'(t)T(u(t)).\end{aligned}$$

By property (ii) of a covariant derivative,

$$\begin{aligned}\frac{D\bar{T}}{dt} &= u''(t)T(u(t)) + u'(t)\frac{DT(u(t))}{dt}\\ &= u''(t)T(u(t)) + u'(t)\frac{DT}{du}\frac{du}{dt} \qquad \text{(by the chain rule, Problem 13.3)}\\ &= u''(t)T(u(t)),\end{aligned}$$

where $DT/du = 0$ because $\gamma(u)$ is a geodesic. Since $T(u)$ has constant length, it is never zero. Therefore,

$$\frac{D\bar{T}}{dt} = 0 \quad\iff\quad u'' = 0 \quad\iff\quad u = \alpha t + \beta$$

for some $\alpha, \beta \in \mathbb{R}$. □

Corollary 14.6. *Let (a,b) be an interval containing 0. For any positive constant $k \in \mathbb{R}^+$, the curve $\gamma(u)$ is a geodesic on (a,b) with initial point q and initial vector v if and only if $\bar{\gamma}(t) := \gamma(kt)$ is a geodesic on $(a/k, b/k)$ with initial point q and initial vector kv.*

Proof. Since

$$\frac{a}{k} < t < \frac{b}{k} \quad\iff\quad a < kt < b,$$

the curve $\bar{\gamma}(t) := \gamma(kt)$ is defined on $(a/k, b/k)$ if and only if $\gamma(t)$ is defined on (a,b). By the chain rule,

$$\bar{\gamma}'(t) = k\gamma'(kt). \tag{14.1}$$

Equivalently,

$$\bar{T}(t) = kT(kt),$$

so that

$$\frac{D\bar{T}}{dt}(t) = k^2\frac{DT}{dt}(kt).$$

Thus, $\bar{\gamma}$ is a geodesic if and only if γ is a geodesic. They have the same initial point $\bar{\gamma}(0) = \gamma(0)$. By (14.1), their initial vectors are related by

$$\bar{\gamma}'(0) = k\gamma'(0).$$

□

14.3 Existence of Geodesics

Suppose M is a manifold with a connection ∇, and $(U,\phi) = (U,x^1,\dots,x^n)$ is a chart relative to which the Christoffel symbols are Γ_{ij}^k. Let $c\colon [a,b] \to M$ be a smooth curve. In the chart (U,ϕ) the curve c has coordinates

$$(\phi \circ c)(t) = (x^1(c(t)),\dots,x^n(c(t))).$$

We will write $y(t) = (\phi \circ c)(t)$ and $y^i(t) = (x^i \circ c)(t)$, and say that c ***is given in*** U ***by***

$$y(t) = (y^1(t),\dots,y^n(t)).$$

In this section we will determine a set of differential equations on $y^i(t)$ for the curve to be a geodesic.

Denote by $\dot{y}^i$ the first derivative dy^i/dt and by $\ddot{y}^i$ the second derivative d^2y^i/dt^2. Let $\partial_i = \partial/\partial x^i$. Then

$$T(t) = c'(t) = \sum \dot{y}^j \partial_{j,c(t)}$$

and

$$\begin{aligned} \frac{DT}{dt}(t) &= \sum_j \ddot{y}^j \partial_{j,c(t)} + \sum_j \dot{y}^j \nabla_{c'(t)} \partial_j \\ &= \sum_j \ddot{y}^j \partial_{j,c(t)} + \sum_{i,j} \dot{y}^j \nabla_{\dot{y}^i \partial_{i,c(t)}} \partial_j. \end{aligned} \tag{14.2}$$

To simplify the notation, we sometimes write ∂_j to mean $\partial_{j,c(t)}$ and Γ_{ij}^k to mean $\Gamma_{ij}^k(c(t))$. Then (14.2) becomes

$$\begin{aligned} \frac{DT}{dt} &= \sum_j \ddot{y}^j \partial_j + \sum_{i,j} \dot{y}^j \nabla_{\dot{y}^i \partial_i} \partial_j \\ &= \sum_k \ddot{y}^k \partial_k + \sum_{i,j,k} \dot{y}^i \dot{y}^j \Gamma_{ij}^k \partial_k \\ &= \sum_k \Big(\ddot{y}^k + \sum_{i,j} \dot{y}^i \dot{y}^j \Gamma_{ij}^k \Big) \partial_k. \end{aligned}$$

So $c(t)$ is a geodesic if and only if

$$\ddot{y}^k + \sum_{i,j} \Gamma_{ij}^k \dot{y}^i \dot{y}^j = 0, \quad k = 1,\dots,n.$$

We summarize this discussion in the following theorem.

Theorem 14.7. *On a manifold with a connection, a parametrized curve $c(t)$ is a geodesic if and only if relative to any chart $(U,\phi) = (U,x^1,\dots,x^n)$, the components of $(\phi \circ c)(t) = (y^1(t),\dots,y^n(t))$ satisfy the system of differential equations*

$$\ddot{y}^k + \sum_{i,j} \Gamma_{ij}^k \dot{y}^i \dot{y}^j = 0, \quad k = 1,\dots,n, \tag{14.3}$$

where the Γ_{ij}^k's are evaluated on $c(t)$.

This is a system of second-order ordinary differential equations on the real line, called the ***geodesic equations***. By an existence and uniqueness theorem of ordinary differential equations, we have the following theorem.

Theorem 14.8. *Let M be a manifold with a connection ∇. Given any point $p \in M$ and tangent vector $v \in T_pM$, there is a geodesic $c(t)$ in M with initial point $c(0) = p$ and initial velocity $c'(0) = v$. Moreover, this geodesic is unique in the sense that any other geodesic satisfying the same initial conditions must agree with $c(t)$ on the intersection of their domains.*

Example 14.9 (*Geodesics on a sphere*). Suppose $\gamma\colon [a,b] \to S^2$ is a geodesic. For $t_0 \in [a,b]$, there is a great circle through $\gamma(t_0)$ with velocity $\gamma'(t_0)$. From Example 14.4, we know that this great circle is a geodesic. Since a geodesic with a given point and a given velocity vector at the point is unique (Theorem 14.8), $\gamma(t)$ must coincide with a great circle wherever it is defined. This proves the converse of Example 14.4 that all geodesics on S^2 lie on great circles.

Let $\gamma_v(t,q)$ be the unique maximal geodesic with initial point q and initial vector $v \in T_qM$. We also write $\gamma_v(t)$. By Corollary 14.6,

$$\gamma_v(kt) = \gamma_{kv}(t)$$

for any positive real number k.

On a Riemannian manifold we always use the unique Riemannian connection to define geodesics. In this case, tangent vectors have lengths and the theory of ordinary differential equations gives the following theorem.

Theorem 14.10. *For any point p of a Riemannian manifold M, there are a neighborhood U of p and numbers $\delta, a > 0$ so that for any $q \in U$ and $v \in T_qM$ with $\|v\| < \delta$, there is a unique geodesic $\gamma\colon (-a,a) \to M$ with $\gamma(0) = q$ and $\gamma'(0) = v$.*

A priori the interval of definition $(-a,a)$ of the geodesic in this theorem may be quite small. However, by rescaling the interval by a constant factor (Corollary 14.6), one can expand the interval to any finite length, at the expense of making the initial vector shorter.

Theorem 14.11. *For any point p of a Riemannian manifold M, there are a neighborhood U of p and a real number $\varepsilon > 0$ so that for any $q \in U$ and $\bar{v} \in T_qM$ with $\|\bar{v}\| < \varepsilon$, there is a unique geodesic $\bar{\gamma}\colon (-2,2) \to M$ with $\bar{\gamma}(0) = q$ and $\bar{\gamma}'(0) = \bar{v}$.*

Proof. Fix $p \in M$ and find a neighborhood U of p and positive numbers δ and a as in Theorem 14.10. Set $k = a/2$. By Corollary 14.6, $\gamma(t)$ is a geodesic on $(-a,a)$ with initial point q and initial vector v if and only if $\bar{\gamma}(t) := \gamma(kt)$ is a geodesic on $(-a/k, a/k) = (-2,2)$ with initial point q and initial vector $\bar{v} := kv$. Moreover,

$$\|v\| = \|\gamma'(0)\| < \delta \quad \iff \quad \|\bar{v}\| = \|\bar{\gamma}'(0)\| = k\|v\| < k\delta = \frac{a\delta}{2}.$$

Choose $\varepsilon = a\delta/2$. Then Theorem 14.10 translates into this one. □

Proposition 14.12. *A connection-preserving diffeomorphism* $f\colon (M,\nabla) \to (\tilde{M},\tilde{\nabla})$ *takes geodesics to geodesics.*

Proof. Suppose $c(t)$ is a geodesic in M, with tangent vector field $T(t) = c'(t)$. If $\tilde{T}(t) = (f\circ c)'(t)$ is the tangent vector field of $f\circ c$, then

$$\tilde{T}(t) = f_*\left(c_*\frac{d}{dt}\right) = f_*(c'(t)) = f_*T.$$

Let D/dt and $\tilde{D}/dt$ be the covariant derivatives along $c(t)$ and $(f\circ c)(t)$, respectively. Then $DT/dt \equiv 0$, because $c(t)$ is a geodesic. By Proposition 13.4,

$$\frac{\tilde{D}\tilde{T}}{dt} = f_*\left(\frac{DT}{dt}\right) = f_*0 = 0.$$

Hence, $f\circ c$ is a geodesic in $\tilde{M}$. □

Because an isometry of Riemannian manifolds preserves the Riemannian connection (Problem 8.2), by Proposition 14.12 it carries geodesics to geodesics.

14.4 Geodesics in the Poincaré Half-Plane

The Poincaré half-plane is covered by a single coordinate open set with coordinates x,y (Figure 14.2). In Example 13.7 we calculated its Christoffel symbols. The

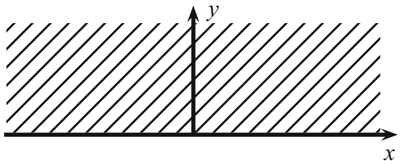


Fig. 14.2. The Poincaré half-plane.

system of geodesic equations (14.3) for the geodesic $c(t) = \big(x(t),y(t)\big)$ consists of two equations:

$$\ddot{x} - \frac{2}{y}\dot{x}\dot{y} = 0, \tag{14.4}$$

$$\ddot{y} + \frac{1}{y}\dot{x}^2 - \frac{1}{y}\dot{y}^2 = 0. \tag{14.5}$$

There is a third equation arising from the fact that a geodesic has constant speed (Proposition 14.3):

$$\langle \dot{c}(t), \dot{c}(t)\rangle = \text{constant}. \tag{14.6}$$

This equation is a consequence of the first two, because any curve satisfying Equations (14.4) and (14.5) will be a geodesic and hence will have constant speed. We include it because of its simplicity. In fact, by a simple reparametrization, we may even assume that the speed is 1. Hence, (14.6) becomes

$$\frac{\dot{x}^2+\dot{y}^2}{y^2}=1$$

or

$$\dot{x}^2+\dot{y}^2=y^2. \tag{14.7}$$

Assuming $\dot{x}$ not identically zero, we can divide (14.4) by $\dot{x}$ and solve it by separation of variables:

$$\frac{\ddot{x}}{\dot{x}}=2\frac{\dot{y}}{y}.$$

Integrating both sides with respect to t gives

$$\ln\dot{x}=2(\ln y)+\text{constant}=\ln(y^2)+\text{constant}.$$

Hence,

$$\dot{x}=ky^2 \tag{14.8}$$

for some constant k

Plugging this into (14.7) and solving for $\dot{y}$ gives

$$k^2y^4+\dot{y}^2=y^2, \tag{14.9}$$

$$\dot{y}=\pm y\sqrt{1-k^2y^2}. \tag{14.10}$$

Dividing (14.8) by (14.10), we get

$$\frac{dx}{dy}=\pm\frac{ky^2}{y\sqrt{1-k^2y^2}}=\pm\frac{ky}{\sqrt{1-k^2y^2}}.$$

Integrate with respect to y to get

$$x=\mp\frac{1}{k}\sqrt{1-k^2y^2}+\text{constant } d.$$

This equation can be put in the form

$$(x-d)^2+y^2=\frac{1}{k^2},$$

which is the equation of a circle centered on the x-axis (Figure 14.3).

In our derivation so far, we assumed that the first derivative $\dot{x}$ is not identically zero. It remains to consider the case $\dot{x}\equiv 0$. If $\dot{x}\equiv 0$, then $x=$ constant, so the curve is a vertical line. Thus, the geodesics in the Poincaré half-plane are either vertical lines or semicircles centered on the x-axis.

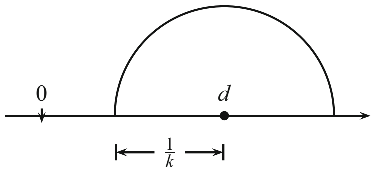


Fig. 14.3. A geodesic in the Poincaré half-plane.

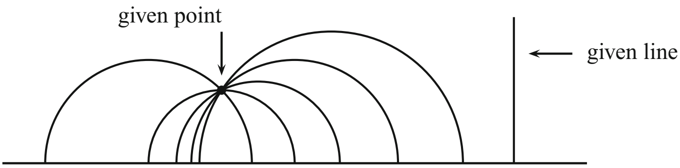


Fig. 14.4. Geodesics in the Poincaré half-plane.

In this calculation we never used (14.5). It is in fact a redundant equation (Problem 14.1).

The Poincaré half-plane is important historically, since it provides a model of a non-Euclidean geometry. In Euclid's axiomatic development of plane geometry, the first four postulates are usually considered self-evident [10, pp. 14–18]. The fifth postulate, called the parallel postulate, was a source of controversy. Two lines in the plane are said to be ***parallel*** if they do not intersect. One form of the fifth postulate states that given a line and a point not on the line, there is a unique parallel line through the given point. For hundreds of years, heroic efforts were made to deduce the fifth postulate from the other four, to derive a contradiction, or to prove its independence. If we interpret the geodesics of the Poincaré half-plane as lines, then the Poincaré half-plane satisfies Euclid's first four postulates, but not the parallel postulate: given a geodesic, say a vertical line, and a point not on the line, there are infinitely many geodesics through the point and not intersecting the given line (see Figure 14.4). In this way the Poincaré half-plane proves conclusively that the parallel postulate is independent of Euclid's first four postulates.

14.5 Parallel Translation

Closely related to geodesics is the notion of parallel translation along a curve. A parallel vector field along a curve is an analogue of a constant vector field in $\mathbb{R}^n$. Throughout the rest of this chapter we assume that M is a manifold with a connection ∇ and $c\colon I \to M$ is a smooth curve in M defined on some interval I.

Definition 14.13. A smooth vector field $V(t)$ along c is ***parallel*** if $DV/dt \equiv 0$ on I.

In this terminology a geodesic is simply a curve c whose tangent vector field $T(t) = c'(t)$ is parallel along c.

Fix a point $p = c(t_0)$ and a frame $e_1, \ldots, e_n$ in a neighborhood U of p in M. Let $[\omega_j^i]$ be the matrix of connection forms of ∇ relative to this frame. If $V(t) = \sum v^i(t) e_{i,c(t)}$ for $c(t) \in U$, then

$$\begin{aligned}
\frac{DV}{dt} &= \sum_i \dot{v}^i e_{i,c(t)} + \sum_i v^i \frac{De_{i,c(t)}}{dt} \\
&= \sum_i \dot{v}^i e_{i,c(t)} + \sum_j v^j \nabla_{c'(t)} e_j \\
&= \sum_i \dot{v}^i e_{i,c(t)} + \sum_{i,j} v^j \omega_j^i(c'(t)) e_{i,c(t)} \\
&= \sum_i \left(\dot{v}^i + \sum_j \omega_j^i(c'(t)) v^j \right) e_{i,c(t)}.
\end{aligned}$$

Thus $DV/dt = 0$ if and only if

$$\dot{v}^i + \sum_{j=1}^n \omega_j^i(c'(t)) v^j = 0, \quad i = 1, \ldots, n. \tag{14.11}$$

This is a system of linear first-order ordinary differential equations. By the existence and uniqueness theorem of ordinary differential equations, there is always a unique solution $V(t)$ on a small interval about t_0 with a given $V(t_0)$. In the next subsection we show that in fact the solution exists not only over a small interval, but also over the entire curve c.

If $V(t)$ is parallel along a curve $c\colon [a,b] \to M$, then we say that $V(b)$ is obtained from $V(a)$ by ***parallel translation*** along c or that $V(b)$ is the ***parallel translate*** or ***parallel transport*** of $V(a)$ along c. By the uniqueness theorem of ordinary differential equations, $V(t)$ is uniquely determined by the initial condition $V(a)$, so if parallel translation exists along c, then it is well defined.

14.6 Existence of Parallel Translation Along a Curve

While a geodesic is guaranteed to exist only locally, parallel translation is possible along the entire length of a curve.

Theorem 14.14. *Let M be a manifold with a connection ∇ and let $c\colon [a,b] \to M$ be a smooth curve in M. Parallel translation is possible from $c(a)$ to $c(b)$ along c, i.e., given a vector $v_0 \in T_{c(a)}$, there exists a parallel vector field $V(t)$ along $c\colon [a,b] \to M$ such that $V(a) = v_0$. Parallel translation along c induces a linear isomorphism $\varphi_{a,b}\colon T_{c(a)}(M) \xrightarrow{\sim} T_{c(b)}(M)$.*

Proof. Because the parallel transport equation $DV/dt = 0$ is $\mathbb{R}$-linear in V, a linear combination with constant coefficients of parallel vector fields along c is again parallel along c.

Let $w_1, \ldots, w_n$ be a basis for $T_{c(t_0)}M$. For each $i = 1, \ldots, n$, there is an interval inside $[a,b]$ of length ε_i about t_0 such that a parallel vector field $W_i(t)$ exists along

$c\colon (t_0-\varepsilon_i, t_0+\varepsilon_i) \to M$ whose value at t_0 is w_i. For ε equal to the minimum of $\varepsilon_1, \dots, \varepsilon_n$, the n basis vectors $w_1, \dots, w_n$ for $T_{c(t_0)}M$ can be parallel translated along c over the interval $(t_0-\varepsilon, t_0+\varepsilon)$. By taking linear combinations as in the remark above, we can parallel translate *every* tangent vector in $T_{c(t_0)}M$ along c over the interval $(t_0-\varepsilon, t_0+\varepsilon)$.

For $t_1 \in (t_0-\varepsilon, t_0+\varepsilon)$ parallel translation along c produces a linear map $\varphi_{t_0,t_1}\colon T_{c(t_0)}M \to T_{c(t_1)}M$. Note that a vector field $V(t)$ is parallel along a curve $c(t)$ if and only if $V(-t)$ is parallel along $c(-t)$, the curve c reparametrized with the opposite orientation. This shows that the linear map φ_{t_0,t_1} has an inverse φ_{t_1,t_0} and is therefore an isomorphism.

Thus for each $t \in [a,b]$ there is an open interval about t over which parallel translation along c is possible. Since $[a,b]$ is compact, it is covered by finitely many such open intervals. Hence, it is possible to parallel translate along c from $c(a)$ to $c(b)$.

□

While a geodesic with a given initial point and initial velocity exists only locally, parallel translation is always possible along the entire length of a smooth curve. In fact, the curve need not even be smooth.

Definition 14.15. A curve $c\colon [a,b] \to M$ is ***piecewise smooth*** if there exist numbers

$$a = t_0 < t_1 < \cdots < t_r = b$$

such that c is smooth on $[t_i, t_{i+1}]$ for $i = 0, \dots, r-1$.

By parallel translating over each smooth segment in succession, one can parallel translate over any piecewise smooth curve.

14.7 Parallel Translation on a Riemannian Manifold

Parallel translation is defined on any manifold with a connection; it is not necessary to have a Riemannian metric. On a Riemannian manifold we will always assume that parallel translation is with respect to the Riemannian connection.

Proposition 14.16. *On a Riemannian manifold M parallel translation preserves length and inner product: if $V(t)$ and $W(t)$ are parallel vector fields along a smooth curve $c\colon [a,b] \to M$, then the length $\|V(t)\|$ and the inner product $\langle V(t), W(t)\rangle$ are constant for all $t \in [a,b]$.*

Proof. Since $\|V(t)\| = \sqrt{\langle V(t), V(t)\rangle}$, it suffices to prove that $\langle V(t), W(t)\rangle$ is constant. By the product rule for the covariant derivative of a connection compatible with the metric (Theorem 13.2),

$$\frac{d}{dt}\langle V, W\rangle = \left\langle \frac{DV}{dt}, W\right\rangle + \left\langle V, \frac{DW}{dt}\right\rangle = 0,$$

since $DV/dt = 0$ and $DW/dt = 0$. Thus $\langle V(t), W(t)\rangle$ is constant as a function of t.

□

Since the angle θ between two vectors u and v can be defined in terms of length and inner product by

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\|v\|}, \quad 0 \le \theta \le \pi,$$

parallel translation also preserves angles.

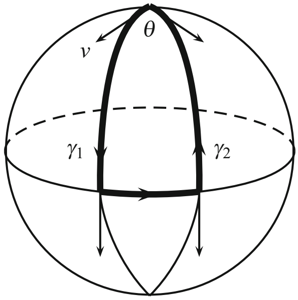


Fig. 14.5. Parallel translating v along a closed piecewise smooth geodesic.

Example 14.17 (*Parallel translation on a sphere*). Let γ_1 and γ_2 be two meridians through the north pole p of a sphere S^2 making an angle θ with each other, and let v be a vector at p tangent to γ_1 (Figure 14.5). As great circles, both γ_1 and γ_2 are geodesics on the sphere. We will parallel translate v along γ_1 to the equator, along the equator to γ_2, and then along γ_2 back to the north pole p.

Since γ_1 is a geodesic, its tangent vector field is parallel, so the parallel translate of v to the equator remains parallel to γ_1 and perpendicular to the equator. Since parallel translation preserves angles, as v is parallel translated along the equator to γ_2, it remains perpendicular to the equator at all times. When it reaches γ_2 along the equator, it will be tangent to γ_2. Parallel translating this vector along γ_2 back to p results in a tangent vector to γ_2 at p. Figure 14.5 shows the position of v at various points of its journey. From this example we see that parallel translating a tangent vector of a manifold around a closed loop need not end in the original vector. This phenomenon is called ***holonomy***.

Problems

14.1.* Geodesic equations
Show that if we differentiate (14.10) with respect to t, we end up with (14.5). Hence, (14.5) is redundant.

14.2. Surface of revolution in $\mathbb{R}^3$
In Problem 5.7 you compute the mean and Gaussian curvatures of a surface of revolution M in $\mathbb{R}^3$.

(a) Using the notations of Problem 5.7, find the geodesic equations of the surface of revolution.
(b) Prove that the meridians of the surface of revolution are geodesics.
(c) Find a necessary and sufficient condition on $f(u)$ and $g(u)$ for a latitude circle to be a geodesic.

14.3. Poincaré disk
Let $\mathbb{D}$ be the Poincaré disk (Problem 11.1).

(a) Show that in polar coordinates (r,θ), the Poincaré metric is given by

$$\langle\ ,\ \rangle_{(r,\theta)} = \frac{4(dr\otimes dr + r^2\, d\theta \otimes d\theta)}{(1-r^2)^2}.$$

Using polar coordinates, compute for $\mathbb{D}$

(b) the Gaussian curvature,
(c) the Christoffel symbols,
(d) the geodesic equations.

§15 Exponential Maps

There are two exponential maps in geometry: the exponential map of a connection and the exponential map for a Lie group. While these are two independent notions, when a Lie group has a bi-invariant Riemannian metric, as all compact Lie groups do, the exponential map for the Lie group coincides with the exponential map of the Riemannian connection.

On a manifold M with a connection, a geodesic is locally defined by a system of second-order ordinary differential equations. By the existence and uniqueness theorems of ordinary differential equations there is a unique geodesic through any given point q with any given direction v. However, this geodesic may be very short, defined only on a small interval $(-a,a)$ about 0. By reparametrizing the geodesic by a constant factor, one can expand the domain of definition of the geodesic at the expense of shortening the initial vector.

On a Riemannian manifold every point p has a neighborhood U in which there is a uniform bound such that all geodesics starting in U with initial vectors of length less than the bound will be defined on an interval including $[-1,1]$ (Theorem 14.11). For any $q \in U$ this leads to the definition of the exponential map Exp_q from a small neighborhood of the origin in T_qM to the manifold M. This exponential map derives its importance from, among other things, providing coordinate charts in which any isometry is represented by a linear map (see Section 15.2).

The exponential map for a Lie group G is defined in terms of the integral curves of the left-invariant vector fields on G. Unlike the exponential map of a connection, the Lie group exponential is defined on the entire Lie algebra $\mathfrak{g}$. It shares some of the properties of the exponential map of a connection. The problems at the end of the chapter explore the relationship between the two notions of exponential map.

15.1 The Exponential Map of a Connection

Suppose M is a manifold with an affine connection. For any point $q \in M$ and vector $v \in T_qM$, denote by $\gamma_v(t,q)$ or simply $\gamma_v(t)$ the unique maximal geodesic with initial point $\gamma_v(0) = q$ and initial vector $\gamma_v'(0) = v$. If $\gamma_v(1)$ is defined, we set

$$\mathrm{Exp}_q(v) = \gamma_v(1).$$

Now assume that M is endowed with a Riemannian metric. For any point $p \in M$, Theorem 14.11 guarantees a neighborhood U of p and a real number $\varepsilon > 0$ such that for all $q \in U$ the exponential map Exp_q is defined on the open ball $B(0,\varepsilon)$ in T_qM:

$$\mathrm{Exp}_q\colon T_qM \supset B(0,\varepsilon) \to M.$$

Let $s\colon M \to TM$ be the zero section of the tangent bundle TM, and let $s(U)$ be the image of $U \subset M$ under s. Viewed as a function of two variables (q,v), the exponential map $\mathrm{Exp}_q(v)$ is defined on an ε-tube around $s(U)$ in TM (see Figure 15.1).

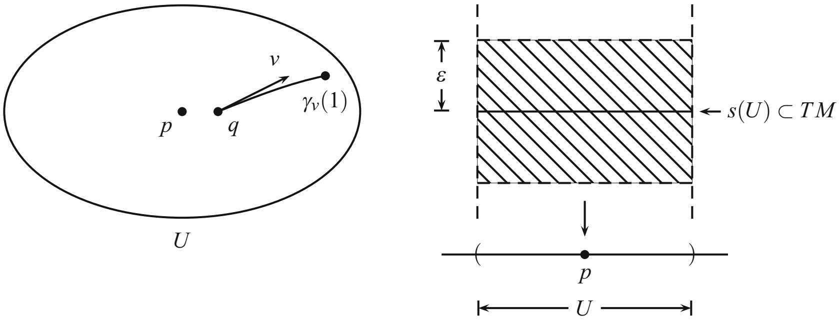


Fig. 15.1. A neighborhood U of p and the domain of the exponential map in the tangent bundle TM.

Proposition 15.1. *On a manifold with a connection the maximal geodesic with initial point q and initial vector v is $\gamma_v(t,q) = \operatorname{Exp}_q(tv)$.*

Proof.

$$\begin{aligned} \operatorname{Exp}_q(tv) &= \gamma_{tv}(1) && \text{(definition of } \operatorname{Exp}_q\text{)} \\ &= \gamma_v(t) && \text{(Corollary 14.6).} \end{aligned}$$

□

Theorem 15.2 (Naturality of the exponential map). *Let $f\colon N \to M$ be an isometry of Riemannian manifolds, and $p \in N$. Suppose $V \subset T_pN$ and $U \subset T_{f(p)}M$ are neighborhoods of the origin on which the exponential maps Exp_p and $\operatorname{Exp}_{f(p)}$ are defined. If the differential $f_{*,p}$ maps V into U, then the following diagram is commutative:*

$$\begin{array}{ccc} V & \xrightarrow{f_{*,p}} & U \\ {\scriptstyle \operatorname{Exp}_p}\downarrow & & \downarrow{\scriptstyle \operatorname{Exp}_{f(p)}} \\ N & \xrightarrow[f]{} & M. \end{array}$$

Proof. For $v \in V \subset T_pN$, let $\gamma_v(t,p)$ be the maximal geodesic in N with initial point p and initial vector v. Since an isometry takes geodesics to geodesics, $(f \circ \gamma_v)(t,p)$ is the maximal geodesic in M with initial point $f(p)$ and initial vector f_*v. By the uniqueness of the geodesic,

$$(f \circ \gamma_v)(t,p) = \gamma_{f_*v}(t, f(p)).$$

Setting $t = 1$ gives

$$f(\operatorname{Exp}_p v) = \operatorname{Exp}_{f(p)}(f_*v).$$

□

15.2 The Differential of the Exponential Map

Suppose p is a point in a Riemannian manifold N and $V \subset T_pN$ is a neighborhood of 0 on which the exponential map Exp_p is defined. Since T_pN is a vector space, its tangent space $T_0(T_pN)$ at the origin can be canonically identified with T_pN.

Proposition 15.3. *The differential* $(\mathrm{Exp}_p)_{*,0}$ *at the origin of the exponential map* $\mathrm{Exp}_p \colon V \subset T_pN \to N$ *is the identity map* $\mathbb{1}_{T_pN} \colon T_pN \to T_pN$.

Proof. For $v \in T_pN$, a curve $c(t)$ in T_pN with $c(0) = 0$ and $c'(0) = v$ is $c(t) = tv$. Using this curve to compute the differential of Exp_p, we have

$$\begin{aligned}(\mathrm{Exp}_p)_{*,0}(v) &= \frac{d}{dt}\bigg|_{t=0} \mathrm{Exp}_p\big(c(t)\big) = \frac{d}{dt}\bigg|_{t=0} \mathrm{Exp}_p(tv) \\ &= \frac{d}{dt}\bigg|_{t=0} \gamma_v(t) \quad \text{(Proposition 15.1)} \\ &= \gamma_v'(0) = v,\end{aligned}$$

where $\gamma_v(t)$ is the unique geodesic with initial point p and initial vector v. □

By the inverse function theorem, the map $\mathrm{Exp}_p \colon V \subset T_pN \to N$ is a local diffeomorphism at $0 \in T_pN$. Thus, there are neighborhoods V' of 0 in T_pN and V'' of p in N such that $\mathrm{Exp}_p \colon V' \to V''$ is a diffeomorphism.

Now suppose $f \colon N \to M$ is an isometry of Riemannian manifolds. As above, there are neighborhoods U' of 0 in $T_{f(p)}M$ and U'' of $f(p)$ in M such that $\mathrm{Exp}_{f(p)} \colon U' \to U''$ is a diffeomorphism. Since $f_{*,p}$ is continuous, one may choose V' sufficiently small so that $f_{*,p}$ maps V' to U'. By the naturality of the exponential map (Theorem 15.2) there is a commutative diagram

$$\begin{array}{ccccccc} T_pN & \supset & V' & \xrightarrow{\ f_{*,p}\ } & U' & \subset & T_{f(p)}M \\ & & \mathrm{Exp}_p \Big\downarrow \simeq & & \simeq \Big\downarrow \mathrm{Exp}_{f(p)} & & \\ N & \supset & V'' & \xrightarrow[\ f\ ]{} & U'' & \subset & M. \end{array}$$

This diagram may be interpreted as follows: relative to the coordinate charts given by the inverse of the exponential map, an isometry of Riemannian manifolds is locally a linear map; more precisely, it is the linear map given by its differential. This is in contrast to the usual statement about the differential $f_{*,p}$ at a point p being the best linear approximation to a C^∞ map in a neighborhood of the point. Here the isometry f is equal to its differential right on the nose.

15.3 Normal Coordinates

Fix a point p in a Riemannian manifold M. By the preceding section, there is a neighborhood V of 0 in T_pM and a neighborhood U of p in M such that the exponential map $\operatorname{Exp}_p\colon V \to U$ is a diffeomorphism. Using the exponential map we can transfer coordinates on T_pM to M. Choose an orthonormal basis $e_1,\dots,e_n$ for T_pM and let $r^1,\dots,r^n$ be the coordinates with respect to the orthonormal basis $e_1,\dots,e_n$ on T_pM. Then $x^1 := r^1 \circ \operatorname{Exp}_p^{-1},\dots,x^n := r^n \circ \operatorname{Exp}_p^{-1}$ is a coordinate system on U such that the tangent vectors $\partial/\partial x^1,\dots,\partial/\partial x^n$ are orthonormal at p. The coordinate neighborhood $(U,x^1,\dots,x^n)$ is called a ***normal neighborhood*** of p and $x^1,\dots,x^n$ are called ***normal coordinates*** on U.

In a normal neighborhood of p, the geodesics through p have a particularly simple expression, for the coordinate expression for the geodesic $\gamma(t) = \operatorname{Exp}_p(at)$ for $a = \sum a^i e_i \in T_pM$ is

$$x(\gamma(t)) = r \circ \operatorname{Exp}_p^{-1}\big(\gamma(t)\big) = at.$$

We write this as $(x^1,\dots,x^n) = (a^1 t,\dots,a^n t)$.

Theorem 15.4. *In a normal neighborhood $(U,x^1,\dots,x^n)$ of p, all the partial derivatives of g_{ij} and all the Christoffel symbols Γ^k_{ij} vanish at p.*

Proof. Let $(x^1,\dots,x^n) = (a^1 t,\dots,a^n t)$ be a geodesic through p. It satisfies the geodesic equations

$$\ddot{x}^k + \sum_{i,j} \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0, \quad k = 1,\dots,n,$$

or

$$\sum \Gamma^k_{ij} a^i a^j = 0.$$

Since this is true for all $(a^1,\dots,a^n)$ at p, setting $(a^1,\dots,a^n) = (0,\dots,1,0,\dots,0,1,\dots,0)$ with $a^i = a^j = 1$ and all other entries 0, we get

$$\Gamma^k_{ij} + \Gamma^k_{ji} = 0.$$

By the symmetry of the connection, $\Gamma^k_{ij} = 0$ at the point p. (At other points, not all of $(x^1,\dots,x^n) = (a^1 t,\dots,a^n t)$ will be geodesics.)

Write ∂_i for $\partial/\partial x^i$. By the compatibility of the connection ∇ with the metric,

$$\partial_k g_{ij} = \partial_k\langle \partial_i, \partial_j\rangle = \langle \nabla_{\partial_k}\partial_i, \partial_j\rangle + \langle \partial_i, \nabla_{\partial_k}\partial_j\rangle.$$

At p, $\nabla_{\partial_k}\partial_i = 0$ and $\nabla_{\partial_k}\partial_j = 0$ since all the Christoffel symbols vanish. Therefore, $(\partial_k g_{ij})(p) = 0$. □

Normal coordinates are especially useful for computation, because at the point p, all $\nabla_{\partial_k}\partial_i = 0$.

15.4 Left-Invariant Vector Fields on a Lie Group

Let G be a Lie group and $\mathfrak{g} = T_eG$ its Lie algebra. Any element $X_e \in \mathfrak{g}$ determines a left-invariant vector field X on G by $X_g := (\ell_g)_*(X_e)$. This establishes a one-to-one correspondence between T_eG and the vector space of left-invariant vector fields on G. Sometimes we adopt an alternate notation for a left-invariant vector field on a Lie group: if $A \in \mathfrak{g}$, then the left-invariant vector field on G generated by A is $\tilde{A}$, with $\tilde{A}_g = \ell_{g*}A$.

Proposition 15.5. *If $\varphi_t(p)$ is an integral curve starting at p of a left-invariant vector field X on a Lie group G, then for any $g \in G$, the left translate $g\varphi_t(p)$ is the integral curve of X starting at gp.*

Proof. We compute the velocity of $g\varphi_t(p)$:

$$\begin{aligned}
\frac{d}{dt}(g\varphi_t(p)) &= (\ell_g)_*\frac{d}{dt}\varphi_t(p) && \text{(definition of } d/dt \text{ and the chain rule)}\\
&= (\ell_g)_*X_{\varphi_t(p)} && (\varphi_t(p) \text{ is an integral curve of } X)\\
&= X_{g\varphi_t(p)} && (X \text{ is left-invariant}).
\end{aligned}$$

This proves that $g\varphi_t(p)$ is also an integral curve of X. At $t = 0$, $g\varphi_t(p) = g\varphi_0(p) = gp$. □

Corollary 15.6. *The local flow φ_t of a left-invariant vector field X on a Lie group G commutes with left multiplication: $\ell_g \circ \varphi_t = \varphi_t \circ \ell_g$ for all $g \in G$, whenever both sides are defined.*

Proof. By Proposition 15.5, both $g\varphi_t(p)$ and $\varphi_t(gp)$ are integral curves of X starting at gp. By the uniqueness of the integral curve, $g\varphi_t(p) = \varphi_t(gp)$. This can be rewritten as $(\ell_g \circ \varphi_t)(p) = (\varphi_t \circ \ell_g)(p)$. □

Proposition 15.7. *The maximal integral curve $\varphi_t(p)$ of a left-invariant vector field X on a Lie group G, where p is a point of G, is defined for all $t \in \mathbb{R}$.*

Proof. By an existence theorem of ordinary differential equations, there are a real number $\varepsilon > 0$ and an integral curve $\varphi_t(e)\colon [-\varepsilon, \varepsilon] \to G$ of X through e. By Proposition 15.5, $p\varphi_t(e) = \varphi_t(p)$ is an integral curve of X defined on $[-\varepsilon, \varepsilon]$ through any point $p \in G$. Let $q = \varphi_\varepsilon(p)$ be the endpoint of $\varphi_t(p)$ on $[-\varepsilon, \varepsilon]$. Then $\varphi_t(q)$ is an integral curve of X defined on $[-\varepsilon, \varepsilon]$. The two integral curves $\varphi_t(p)$ and $\varphi_t(q)$ agree on their overlap and so the domain of $\varphi_t(p)$ can be extended to $[-\varepsilon, 2\varepsilon]$. Thus, the domain of an integral curve of X through any point can always be extended an extra ε unit. By induction, the maximal integral curve of X through any point is defined for all $t \in \mathbb{R}$. □

It follows from this proposition that a left-invariant vector field X on a Lie group G has a ***global flow*** $\varphi\colon \mathbb{R} \times G \to G$.

15.5 Exponential Map for a Lie Group

For a Lie group G there is also a notion of an exponential map. Unlike the exponential map of a connection, which is a priori defined only on a small neighborhood of 0 in the tangent space T_pM of each point p in a manifold with a connection, the exponential map for a Lie group is defined on the entire tangent space T_eG, but only for the tangent space at the identity. When the Lie group has a bi-invariant Riemannian metric (Section 15.8), the exponential map for the Lie group coincides with the exponential map of the Riemannian connection.

For $X_e \in \mathfrak{g} := T_eG$, denote by X the left-invariant vector field on G generated by X_e. In this section we consider primarily integral curves starting at the identity e. To show the dependence on X, we write $c_X(t) = \varphi_t(e)$ for the integral curve through e of the left-invariant vector field X on G.

If s is a real number, then

$$\begin{aligned}\frac{d}{dt}c_X(st) &= sc_X'(st) \quad \text{(chain rule)}\\ &= sX_{c_X(st)} \quad (c_X \text{ is an integral curve of } X).\end{aligned}$$

This shows that $c_X(st)$ as a function of t is an integral curve through e of the left-invariant vector field sX. Hence,

$$c_X(st) = c_{sX}(t). \tag{15.1}$$

Definition 15.8. The ***exponential map*** for a Lie group G with Lie algebra $\mathfrak{g}$ is the map $\exp\colon \mathfrak{g} \to G$ defined by $\exp(X_e) = c_X(1)$ for $X_e \in \mathfrak{g}$.

To distinguish the exponential map of a connection from the exponential map for a Lie group, we denote the former as Exp_p and the latter as exp.

Proposition 15.9. (i) *For $X_e \in \mathfrak{g}$, the integral curve starting at e of the left-invariant vector field X is* $\exp(tX_e) = c_X(t) = \varphi_t(e)$.
(ii) *For $X_e \in \mathfrak{g}$ and $g \in G$, the integral curve starting at g of the left-invariant vector field X is $g\exp(tX_e)$.*
(iii) *For $s,t \in \mathbb{R}$ and $X_e \in \mathfrak{g}$,* $\exp\big((s+t)X_e\big) = (\exp sX_e)(\exp tX_e)$.
(iv) *The exponential map* $\exp\colon \mathfrak{g} \to G$ *is a C^∞ map.*
(v) *The differential at* 0 *of the exponential map,* $\exp_{*,0}\colon T_0(\mathfrak{g}) = \mathfrak{g} \to T_eG = \mathfrak{g}$ *is the identity map.*
(vi) *For the general linear group* $\mathrm{GL}(n,\mathbb{R})$,

$$\exp A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \quad \text{for } A \in \mathfrak{gl}(n,\mathbb{R}).$$

Proof. (i) By the definition of exp and by (15.1),

$$\exp(tX_e) = c_{tX}(1) = c_X(t).$$

(ii) This follows from (i) and Proposition 15.5.
(iii) View s as a constant and differentiate with respect to t:

$$\frac{d}{dt}c_X(s+t) = c_X'(s+t) = X_{c_X(s+t)}.$$

This shows that $c_X(s+t)$ as a function of t is an integral curve with initial point $c_X(s)$ of the left-invariant vector field X. With s fixed, $c_X(s)c_X(t)$ is also an integral curve of X with initial point $c_X(s)$. By the uniqueness of integral curves,

$$c_X(s+t) = c_X(s)c_X(t).$$

Using (i), we can rewrite this equation as $\exp((s+t)X_e) = (\exp sX_e)(\exp tX_e)$.
(iv) (following [22], p. 103) The proof is based on the fact from the theory of ordinary differential equations that the flow of a C^∞ vector field is C^∞. Recall that for each $X_e \in \mathfrak{g}$, X is the left-invariant vector field generated by X_e. Define a vector field V on $G \times \mathfrak{g}$ by

$$V_{(g,X_e)} = (X_g, 0) = (\ell_{g*}(X_e), 0).$$

By (ii), the integral curve of V starting at (g, X_e) is $c(t) = (g \exp tX_e, X_e)$. The global flow of V is

$$\begin{aligned} \varphi \colon \mathbb{R} \times (G \times \mathfrak{g}) &\to G \times \mathfrak{g}, \\ \varphi(t, (g, X_e)) &= (g \exp tX_e, X_e). \end{aligned}$$

Let $\pi \colon G \times \mathfrak{g} \to G$ be the projection to the first factor. Then

$$\exp(X_e) = (\pi \circ \varphi)\big(1, (e, X_e)\big).$$

As a composite of C^∞ maps, $\exp \colon \mathfrak{g} \to G$ is C^∞.
(v) For $X_e \in \mathfrak{g}$, a C^∞ curve starting at 0 in $\mathfrak{g}$ with initial vector X_e is $c(t) = tX_e$. Computing the differential $\exp_{*,0}$ using this curve, we get by (i) that

$$\exp_{*,0}(X_e) = \left.\frac{d}{dt}\right|_{t=0} \exp c(t) = \left.\frac{d}{dt}\right|_{t=0} \exp(tX_e) = X_e.$$

(vi) We know from ([21], Sect. 15.3, p. 170) that $c(t) = \sum_{k=0}^\infty A^k t^k/k!$ is absolutely convergent for $A \in \mathfrak{gl}(n, \mathbb{R})$. Denote by $\tilde{A}$ the left-invariant vector field on $\mathrm{GL}(n, \mathbb{R})$ generated by A. By [21, Example 8.19],

$$\tilde{A}_g = (\ell_g)_* A = gA.$$

Now $c(0) = I$, the identity matrix, and

$$\begin{aligned} c'(t) &= \sum_{k=1}^\infty \frac{kA^k t^{k-1}}{k!} \\ &= \sum_{k=1}^\infty \frac{A^{k-1} t^{k-1}}{(k-1)!} A \\ &= c(t)A = \tilde{A}_{c(t)}. \end{aligned}$$

Hence $c(t)$ is the integral curve of $\tilde{A}$ through I. By (i),

$$\exp tA = c(t) = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}.$$

Setting $t = 1$ gives the desired identity. □

Definition 15.10. A 1-***parameter subgroup*** of a Lie group G is a group homomorphism $\varphi \colon \mathbb{R} \to G$.

In this terminology, Proposition 15.9(iii) says that if X is a left-invariant vector field on a Lie group G, then its integral curve $c_X(t) = \exp(tX_e)$ through the identity e is a 1-parameter subgroup of G.

Corollary 15.11. *The diffeomorphism $\varphi_t \colon G \to G$ induced from the flow of a left-invariant vector field X on a Lie group G is right multiplication by $\exp tX_e$:*

$$\varphi_t(g) = r_{\exp tX_e}(g).$$

Proof. Applying Corollary 15.6 to e, we get

$$\varphi_t(g) = \varphi_t(ge) = g\varphi_t(e) = r_{\varphi_t(e)}(g) = r_{\exp tX_e}(g). \qquad \square$$

15.6 Naturality of the Exponential Map for a Lie Group

Just as the exponential map for a Riemannian manifold is natural with respect to isometries, so the exponential map for a Lie group is natural with respect to Lie group homomorphisms.

Theorem 15.12. *Let H and G be Lie groups with Lie algebras $\mathfrak{h}$ and $\mathfrak{g}$, respectively. If $f \colon H \to G$ is a Lie group homomorphism, then the diagram*

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{f_*} & \mathfrak{g} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ H & \xrightarrow[f]{} & G \end{array}$$

commutes.

Lemma 15.13. *If $f \colon H \to G$ is a group homomorphism, then for any $h \in H$,*

$$f \circ \ell_h = \ell_{f(h)} \circ f. \tag{15.2}$$

Proof (of lemma). For $x \in H$,

$$(f \circ \ell_h)(x) = f(hx) = f(h)f(x) = (\ell_{f(h)} \circ f)(x). \qquad \square$$

Proof (*of theorem*). For $A \in \mathfrak{h}$, denote by $\tilde{A}$ the left-invariant vector field on the Lie group H generated by A. If $c(t)$ is an integral curve of $\tilde{A}$ through e, then

$$c'(t) = \tilde{A}_{c(t)} = (\ell_{c(t)})_* A \tag{15.3}$$

We will now show that $(f \circ c)(t)$ is an integral curve of $\widetilde{f_* A}$, the left-invariant vector field on G generated by $f_* A \in \mathfrak{g}$. Taking the differential of (15.2), we get

$$f_* \circ \ell_{h*} = \ell_{f(h)*} \circ f_*. \tag{15.4}$$

Then

$$\begin{aligned}
(f \circ c)'(t) &= f_* c'(t) && \text{(by the chain rule again)} \\
&= f_* \ell_{c(t)*} A && \text{(by (15.3))} \\
&= \ell_{f(c(t))*}(f_* A) && \text{(by (15.4))} \\
&= \widetilde{f_* A}_{(f \circ c)(t)}.
\end{aligned}$$

By the definition of the exponential map for a Lie group,

$$\exp(f_* A) = (f \circ c)(1) = f(c(1)) = f(\exp A). \qquad \square$$

15.7 Adjoint Representation

Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $g \in G$, define $c_g \colon G \to G$ to be ***conjugation by*** g:

$$c_g(x) = gxg^{-1}.$$

The differential at the identity of c_g is denoted by

$$\mathrm{Ad}(g) = (c_g)_{*,e} \colon \mathfrak{g} \to \mathfrak{g}.$$

For $g, h \in G$,

$$c_{gh} = c_g \circ c_h,$$

so that by the chain rule,

$$(c_{gh})_* = (c_g)_* \circ (c_h)_*,$$

or

$$\mathrm{Ad}(gh) = \mathrm{Ad}(g) \circ \mathrm{Ad}(h).$$

Thus, $\mathrm{Ad} \colon G \to \mathrm{GL}(\mathfrak{g})$ is a group homomorphism; it is called the ***adjoint representation*** of the Lie group G.

Proposition 15.14. *Let G be a Lie group. The adjoint representation* $\mathrm{Ad} \colon G \to \mathrm{GL}(\mathfrak{g})$ *is a C^∞ map.*

Proof. By the definition of a Lie group, the Lie group multiplication $\mu\colon G\times G\to G$ and inversion $g\mapsto g^{-1}$ are C^∞ maps. In terms of μ,

$$c_g(x)=\mu(g,\mu(x,g^{-1})),\quad g,x\in G,$$

which shows that as a function of g and x, the conjugation $c_g(x)$ is also C^∞.

Let $x^1,\dots,x^n$ be a coordinate system in a neighborhood of the identity e. Then $\mathrm{Ad}(g)=(c_g)_{*,e}$ is represented by the Jacobian matrix $[(\partial(x^i\circ c_g)/\partial x^j)(e)]$. Since all the partial derivatives $\partial(x^i\circ c_g)/\partial x^j(e)$ are C^∞ in g, $\mathrm{Ad}(g)$ is a C^∞ function of g. □

The differential of Ad at the identity is a Lie algebra homomorphism

$$\mathrm{ad}:=(\mathrm{Ad})_{*,e}\colon\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g}),$$

called the ***adjoint representation of the Lie algebra*** $\mathfrak{g}$. We usually write $\mathrm{ad}(A)(B)$ as $\mathrm{ad}_A B$.

Proposition 15.15. *For $A,B\in\mathfrak{g}$, $\mathrm{ad}_A B=[A,B]$.*

Proof. For $A\in\mathfrak{g}$, let $\tilde{A}$ be the left-invariant vector field generated by A. We will write the exponential $\exp(A)$ as e^A. By Proposition 15.9(i), the integral curve through e of $\tilde{A}$ is $\varphi_t(e)=e^{tA}$, and by Corollary 15.11 the diffeomorphism $\varphi_t\colon G\to G$ induced from the flow of $\tilde{A}$ is right multiplication by e^{tA}. For $A,B\in\mathfrak{g}$,

$$\begin{aligned}
\mathrm{ad}_A B &= (\mathrm{Ad}_{*,e}A)(B) && \text{(definition of ad)}\\
&= \frac{d}{dt}\bigg|_{t=0}\mathrm{Ad}(e^{tA})(B) && \text{(computing }\mathrm{Ad}_*\text{ using the curve }\varphi_t(e)=e^{tA})\\
&= \frac{d}{dt}\bigg|_{t=0}(c_{e^{tA}})_*(B) && \text{(definition of Ad)}\\
&= \frac{d}{dt}\bigg|_{t=0}(r_{e^{-tA}})_*(\ell_{e^{tA}})_*B && \text{(definition of conjugation }c(e^{tA}))\\
&= \frac{d}{dt}\bigg|_{t=0}(r_{e^{-tA}})_*\tilde{B}_{e^{tA}} && \text{(definition of }\tilde{B})\\
&= \frac{d}{dt}\bigg|_{t=0}(\varphi_{-t})_*(\tilde{B}_{e^{tA}}) && (\varphi_t=r_{e^{tA}},\ \varphi_t(e)=e^{tA})\\
&= (\mathcal{L}_{\tilde{A}}\tilde{B})_e && \text{(definition of the Lie derivative }\mathcal{L}_{\tilde{A}}\tilde{B})\\
&= [\tilde{A},\tilde{B}]_e && \text{([21, Th. 20.4, p. 225])}\\
&= [A,B] && \text{(definition of }[A,B]).
\end{aligned}$$

□

15.8 Associativity of a Bi-Invariant Metric on a Lie Group

A Riemannian metric $\langle\ ,\ \rangle$ on a Lie group G is said to be ***left-invariant*** if for all C^∞ vector fields X,Y on G and $g\in G$,

$$\langle\ell_{g*}X,\ell_{g*}Y\rangle=\langle X,Y\rangle.$$

A ***right-invariant metric*** is defined similarly with r_g in place of ℓ_g. A metric that is both left- and right-invariant is said to be ***bi-invariant***.

Proposition 15.16. *If $\langle\ ,\ \rangle$ is a bi-invariant metric on a Lie group G, then for all left-invariant vector fields X, Y, Z on G,*

$$\langle [X,Y], Z\rangle = \langle X, [Y,Z]\rangle.$$

Proof. Since $\langle\ ,\ \rangle$ is bi-invariant, for any $g \in G$ and $X_e, Z_e \in \mathfrak{g}$,

$$\begin{aligned}\langle X_e, Z_e\rangle &= \langle \ell_{g*} r_{g^{-1}*} X_e, \ell_{g*} r_{g^{-1}*} Z_e\rangle \\ &= \langle (\operatorname{Ad} g)X_e, (\operatorname{Ad} g)Z_e\rangle.\end{aligned}$$

If $c(t)$ is the integral curve through e of the left-invariant vector field Y, then

$$\langle X_e, Z_e\rangle = \langle \big(\operatorname{Ad} c(t)\big)X_e, \big(\operatorname{Ad} c(t)\big)Z_e\rangle \quad \text{for all } t. \tag{15.5}$$

Differentiating (15.5) with respect to t and evaluating at $t = 0$ gives

$$\begin{aligned}0 &= \frac{d}{dt}\bigg|_{t=0} \langle \big(\operatorname{Ad} c(t)\big)X_e, \big(\operatorname{Ad} c(t)\big)Z_e\rangle \\ &= \langle \big(\operatorname{Ad}_* c'(0)\big)X_e, \big(\operatorname{Ad} c(0)\big)Z_e\rangle + \langle \big(\operatorname{Ad} c(0)\big)X_e, \big(\operatorname{Ad}_* c'(0)\big)Z_e\rangle \\ &= \langle \operatorname{ad}_{Y_e} X_e, Z_e\rangle + \langle X_e, \operatorname{ad}_{Y_e} Z_e\rangle \\ &= \langle [Y_e, X_e], Z_e\rangle + \langle X_e, [Y_e, Z_e]\rangle \quad \text{(by Proposition 15.15)} \\ &= -\langle [X_e, Y_e], Z_e\rangle + \langle X_e, [Y_e, Z_e]\rangle.\end{aligned}$$

Since $\langle\ ,\ \rangle$ is left-invariant and $\ell_{g,*}[X_e, Y_e] = [\ell_{g,*}(X_e), \ell_{g,*}(Y_e)] = [X_g, Y_g]$ for all $g \in G$ (see [21, Proposition 16.14, p. 185]), left-translating the equation above by g gives the proposition. □

Problems

15.1. Exponential maps on $\mathbb{R}^2$
Give $\mathbb{R}^2$ its usual Euclidean metric.

(a) At any point $p \in \mathbb{R}^2$, the tangent space $T_p\mathbb{R}^2$ can be canonically identified with $\mathbb{R}^2$. Show that under this identification $T_p\mathbb{R}^2 \simeq \mathbb{R}^2$, the exponential map $\operatorname{Exp}_p \colon T_p\mathbb{R}^2 \to \mathbb{R}^2$ is the identity map.
(b) Viewed as a Lie group under addition, $\mathbb{R}^2$ has Lie algebra $\mathbb{R}^2$. Show that the exponential map $\exp\colon \mathbb{R}^2 \to \mathbb{R}^2$ is also the identity map.

15.2. Exponentials as generators
Prove that a connected Lie group G with Lie algebra $\mathfrak{g}$ is generated by $\exp A$ for all $A \in \mathfrak{g}$; i.e., every element $g \in G$ is a product of finitely many exponentials: $g = (\exp A_1)\cdots(\exp A_r)$.

15.3. Invariant Riemannian metrics on a Lie group
Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Suppose $\langle\ ,\ \rangle_e$ is an inner product on $\mathfrak{g}$. For any $g \in G$ and $X_g, Y_g \in T_gG$, define

$$\langle X_g, Y_g\rangle = \langle \ell_{g^{-1}*} X_g, \ell_{g^{-1}*} Y_g\rangle_e.$$

(a) Show that $\langle\ ,\ \rangle$ is a left-invariant Riemannian metric on G.
(b) Show that this sets up a bijection

$$\{\text{inner products on } \mathfrak{g}\} \longleftrightarrow \{\text{left-invariant Riemannian metrics on } G\}.$$

(c) Show that under the bijection in (b), $\mathrm{Ad}(G)$-invariant inner products on $\mathfrak{g}$ correspond to bi-invariant Riemannian metrics on G.

15.4. Riemannian connection on a Lie group with a bi-invariant metric
Let G be a Lie group with a bi-invariant Riemannian metric. Prove that if ∇ is its Riemannian connection, then $\nabla_X Y = \frac{1}{2}[X,Y]$ for all left-invariant vector fields X and Y on G. (*Hint*: Use (6.8) and Proposition 15.16.)

15.5. Geodesics on a Lie group with a bi-invariant metric

(a) Let G be a Lie group with a bi-invariant Riemannian metric. Prove that the geodesics on G are precisely the integral curves of left-invariant vector fields. (*Hint*: Let $c(t)$ be an integral curve through e of a left-invariant vector field X. Apply Problem 15.4 to show that $c(t)$ is a geodesic.)
(b) Show that if a Lie group has a bi-invariant Riemannian metric, then the exponential map for the Lie group coincides with the exponential map Exp_e of the Riemannian connection.

15.9 Addendum. The Exponential Map as a Natural Transformation

The naturality of the exponential map for a Riemannian manifold says in fact that the exponential map is a natural transformation between two functors. There are two functors lurking in the background in Theorem 15.2, and in this addendum we will ferret them out.

Let F and G be two functors from a category $\mathcal{A}$ to a category $\mathcal{B}$. A **natural transformation** from F to G is a collection of morphisms $\varphi_A\colon F(A) \to G(A)$ in $\mathcal{B}$, one for each object A in $\mathcal{A}$, such that for each morphism $f\colon A \to A'$ in $\mathcal{A}$ the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ {\scriptstyle\varphi_A}\downarrow & & \downarrow{\scriptstyle\varphi_{A'}} \\ G(A) & \xrightarrow[G(f)]{} & G(A') \end{array}$$

is commutative.

Let $\mathcal{A}$ be the category in which the objects are pointed Riemannian manifolds (M,p) and for any two objects (N,q) and (M,p), a morphism $(N,q) \to (M,p)$ is an isometry $f\colon N \to M$ such that $f(q) = p$.

Next, we define the category $\mathcal{B}$. Consider the set of all pairs (U,p), where U is an open subset of a manifold M and $p \in U$. We say that two pairs (U,p) and (V,q) are ***equivalent***, written $(U,p) \sim (V,q)$, if $p = q$ and U and V are open subsets of the same manifold. This is clearly an equivalence relation. An equivalence class of such

pairs (U,p) with $U \subset M$ is called a **germ of neighborhoods** of p in M. A morphism $f\colon [(U,p)] \to [(V,q)]$ from the germ of (U,p) to the germ of (V,q) is a collection of C^∞ maps $f_{U',V'}\colon (U',p) \to (V',q)$ such that

$$(U',p) \sim (U,p), \quad (V',q) \sim (V,q),$$

and for any overlapping (U',p) and (U'',p) on which $f_{U',V'}$ and $f_{U'',V''}$ are defined,

$$f_{U',V'}\big|_{U'\cap U''} = f_{U'',V''}\big|_{U'\cap U''}.$$

This collection $\{f_{U',V'}\}$ need not be defined on all $(U',p) \in [(U,p)]$; it is enough that $f_{U',V'}$ be defined on some neighborhood U' of p in M. As the category $\mathcal{B}$, we take the objects to be all germs of neighborhoods and the morphisms to be morphisms of germs of neighborhoods.

The functor $F\colon \mathcal{A} \to \mathcal{B}$ takes a pointed Riemannian manifold (M,p) to the germ $[(T_pM,0)]$ of neighborhoods of the origin in the tangent space T_pM, and takes an isometry $f\colon (N,q) \to (M,p)$ of pointed Riemannian manifolds to the morphism of germs of neighborhoods induced by the differential $f_{*,q}\colon (T_qN,0) \to (T_pM,0)$.

The functor $G\colon \mathcal{A} \to \mathcal{B}$ takes a pointed Riemannian manifold (M,p) to its germ $[(M,p)]$ of neighborhoods, and takes an isometry $f\colon (N,q) \to (M,p)$ to the morphism of germs of neighborhoods induced by f.

The exponential map Exp_q of a Riemannian manifold N at q induces a morphism $\mathrm{Exp}_q\colon [(T_qN,0)] \to [(N,q)]$ of germs of neighborhoods. In the language of categories and functors, Theorem 15.2 is equivalent to the statement that the exponential map is a natural transformation from the functor F to the functor G.

§16 Distance and Volume

In differential topology, where two manifolds are considered the same if they are diffeomorphic, the concepts of length and volume do not make sense, for length and volume are clearly not invariant under diffeomorphisms. In differential geometry, where two Riemannian manifolds are considered the same if they are isometric, length and volume do make sense, for they are invariant under isometries. In this chapter we define length and distance on a Riemannian manifold and construct a volume form on an oriented Riemannian manifold.

16.1 Distance in a Riemannian Manifold

In a Riemannian manifold we defined the ***length*** of a tangent vector $X_p \in T_pM$ to be

$$\|X_p\| = \sqrt{\langle X_p, X_p \rangle}.$$

If $c\colon [a,b] \to M$ is a parametrized curve in M, its ***arc length*** is given by

$$\ell(c) = \int_a^b \|c'(t)\| dt.$$

We can reparametrize $c(t)$ via a diffeomorphism $t(u)\colon [a_1,b_1] \to [a,b]$; the ***reparametrization*** of $c(t)$ is then $c(t(u))\colon [a_1,b_1] \to M$. We say that the reparametrization is ***orientation-preserving*** if it preserves the order of the endpoints: $t(a_1) = a$ and $t(b_1) = b$; it is ***orientation-reversing*** if it reverses the order of the endpoints. Since a diffeomorphism $t(u)\colon [a_1,b_1] \to [a,b]$ is one-to-one, it is either increasing or decreasing. Let $u(t)$ be its inverse. Because $u(t(u)) = u$, by the chain rule,

$$\frac{dt}{du} \cdot \frac{du}{dt} = 1,$$

which shows that dt/du is never zero. So dt/du is either always positive or always negative, depending on whether the reparametrization $t(u)$ preserves or reverses the orientation.

Proposition 16.1. *The arc length of a curve $c\colon [a,b] \to M$ in a Riemannian manifold M is independent of its parametrization.*

Proof. Suppose $c(t(u)), a_1 \le u \le b_1$ is an orientation-preserving reparametrization. Then $dt/du > 0$. By the chain rule,

$$\frac{d}{du}c(t(u)) = c'(t(u))\frac{dt}{du}. \tag{16.1}$$

It follows that

$$\begin{aligned}
\int_{a_1}^{b_1} \left\| \frac{d}{du} c(t(u)) \right\| du &= \int_{a_1}^{b_1} \|c'(t(u))\| \left| \frac{dt}{du} \right| du \\
&= \int_{a_1}^{b_1} \|c'(t(u))\| \frac{dt}{du}\, du \\
&= \int_a^b \|c'(t)\|\, dt \quad \text{(change of variables formula).}
\end{aligned}$$

If $c(t(u))$ is an orientation-reversing parametrization, the calculation is the same except that (i) $|dt/du| = -dt/du$, because $dt/du < 0$, and (ii) the limits of integration in the last line are reversed. These two changes cancel out:

$$\begin{aligned}
\int_{a_1}^{b_1} \left\| \frac{d}{du} c\big(t(u)\big) \right\| du &= \int_{a_1}^{b_1} \|c'\big(t(u)\big)\| \left| \frac{dt}{du} \right| du \\
&= -\int_{a_1}^{b_1} \|c'\big(t(u)\big)\| \frac{dt}{du}\, du = \int_{b_1}^{a_1} \|c'\big(t(u)\big)\| \frac{dt}{du}\, du \\
&= \int_a^b \|c'(t)\|\, dt \quad \text{(change of variables formula).}
\end{aligned}$$

So the arc length is still independent of the parametrization. □

Given two points p and q on a connected Riemannian manifold, we define the ***distance*** between them to be

$$d(p,q) = \inf_c \ell(c),$$

where the infimum is taken over all piecewise smooth curves c from p to q and $\ell(c)$ is the length of the curve c.

For the distance $d(p,q)$ to be defined, there must be a curve joining p and q. Hence, in order to have a distance function, the Riemannian manifold should be path-connected. By a theorem of general topology, a locally path-connected space is path-connected if and only if it is connected (Problem A.2). Being locally Euclidean, a manifold is locally path-connected. Thus, for a manifold, path-connectedness is equivalent to connectedness. Whenever we speak of the distance function on a Riemannian manifold M, we will assume M to be connected.

It is easily verified that the distance function on a Riemannian manifold satisfies the three defining properties of a metric on a metric space: for all $p,q,r \in M$,

(i) positive-definiteness: $d(p,q) \geq 0$ and equality holds if and only if $p = q$;
(ii) symmetry: $d(p,q) = d(q,p)$;
(iii) triangle inequality: $d(p,r) \leq d(p,q) + d(q,r)$.

Therefore, with the distance as a metric, the Riemannian manifold M becomes a metric space.

Remark 16.2. This metric is not the Riemannian metric, which is an inner product on the tangent space T_pM at each point p of the manifold. In the literature a "Riemannian metric" is sometimes shortened to a "metric." So the word "metric" can have two different meanings in Riemannian geometry.

Example 16.3. Let M be the punctured plane $\mathbb{R}^2 - \{(0,0)\}$ with the Euclidean metric as the Riemannian metric. There is no geodesic from $(-1,-1)$ to $(1,1)$, because the origin is missing from the manifold, but the distance between the two points is defined and is equal to $2\sqrt{2}$ (Figure 16.1). This example shows that the distance between two points need not be realized by a geodesic.

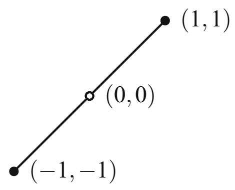


Fig. 16.1. Punctured plane.

16.2 Geodesic Completeness

In differential topology, a curve $c\colon (a,b) \to M$ in a manifold M can be reparametrized so that its domain is $\mathbb{R}$, because the interval (a,b) is diffeomorphic to $\mathbb{R}$. Now suppose the manifold M is Riemannian and the curve is a geodesic. By Proposition 14.5, any reparametrization of the geodesic which keeps it a geodesic must have the form $t = \alpha u + \beta$, for $\alpha, \beta \in \mathbb{R}$. Such a linear change of variables maps a finite interval to another finite interval. Thus, whether or not a geodesic has the entire real line $\mathbb{R}$ as its domain is independent of the parametrization.

Definition 16.4. A Riemannian manifold is said to be ***geodesically complete*** if the domain of every geodesic in it can be extended to $\mathbb{R}$.

Example. The Euclidean space $\mathbb{R}^n$ is geodesically complete.

Example. The sphere S^2 is geodesically complete.

Example. The punctured Euclidean plane $\mathbb{R}^2 - \{(0,0)\}$ is not geodesically complete, since the domain of the geodesic $c(t) = (t,t)$, $1 < t < \infty$ cannot be extended to $-\infty < t < \infty$.

Definition 16.5. A geodesic defined on $[a,b]$ is said to be ***minimal*** if its length is minimal among all piecewise smooth curves joining its two endpoints.

A proof of the following fundamental theorem on geodesic completeness may be found in [19, Volume 1, Theorem 18, p. 342].



Heinz Hopf (1894–1971) and Willi Rinow (1907–1979)

Theorem 16.6 (Hopf–Rinow theorem). *A Riemannian manifold is geodesically complete if and only if it is complete as a metric space in the distance metric d defined above. Moreover, in a geodesically complete Riemannian manifold, any two points may be joined by a minimal geodesic.*

16.3 Dual 1-Forms Under a Change of Frame

Many of the constructions in differential geometry are local, in terms of a frame of vector fields and its dual frame of 1-forms, for example, the connection matrix and the curvature matrix of an affine connection. To see how the construction transforms under a change of frame, it is useful to know how the dual frame of 1-forms transforms.

If $e_1, \ldots, e_n$ and $\bar{e}_1, \ldots, \bar{e}_n$ are two C^∞ frames on an open set U in a manifold M, then

$$\bar{e}_j = \sum_i a^i_j e_i \tag{16.2}$$

for some C^∞ function $[a^i_j] \colon U \to \mathrm{GL}(n, \mathbb{R})$. Let $\theta^1, \ldots, \theta^n$ and $\bar{\theta}^1, \ldots, \bar{\theta}^n$ be the dual frames of 1-forms, respectively, meaning

$$\theta^i(e_j) = \delta^i_j \quad \text{and} \quad \bar{\theta}^i(\bar{e}_j) = \delta^i_j,$$

where δ^i_j is the Kronecker delta. To write $\bar{\theta}^1, \ldots, \bar{\theta}^n$ in terms of $\theta^1, \ldots, \theta^n$, it is best to use matrix notation.

In matrix notation we write the frame $e_1, \ldots, e_n$ as a row vector

$$e = [e_1 \ \cdots \ e_n]$$

and the dual frame $\theta^1,\dots,\theta^n$ as a column vector

$$\theta = \begin{bmatrix} \theta^1 \\ \vdots \\ \theta^n \end{bmatrix}.$$

The duality is expressed in matrix notation as a matrix product:

$$\theta e = \begin{bmatrix} \theta^1 \\ \vdots \\ \theta^n \end{bmatrix} [e_1 \ \cdots \ e_n] = [\theta^i(e_j)] = [\delta^i_j] = I.$$

Equation (16.2) translates into

$$\bar{e} = [\bar{e}_1 \ \cdots \ \bar{e}_n] = [e_1 \ \cdots \ e_n][a^i_j] = eA.$$

Proposition 16.7. *Let $e = [e_1 \ \cdots \ e_n]$ and $\bar{e} = [\bar{e}_1 \ \cdots \ \bar{e}_n]$ be two frames on an open set U. If $\bar{e} = eA$ for $A\colon U \to \mathrm{GL}(n,\mathbb{R})$, then the dual frames $\bar{\theta}$ and θ are related by $\bar{\theta} = A^{-1}\theta$.*

What this means is that if $\bar{\theta}^i = \sum b^i_j \theta^j$, then $B = [b^i_j] = A^{-1}$.

Proof. Since

$$(A^{-1}\theta)\bar{e} = A^{-1}\theta e A = A^{-1} I A = I,$$

we have $\bar{\theta} = A^{-1}\theta$. □

Next we study how $\bar{\theta}^1 \wedge \cdots \wedge \bar{\theta}^n$ is related to $\theta^1 \wedge \cdots \wedge \theta^n$.

Proposition 16.8 (Wedge product under a change of frame). *Let V be a vector space of dimension n and $\theta^1,\dots,\theta^n$ a basis for the dual space $V^\vee$. If the 1-covectors $\bar{\theta}^1,\dots,\bar{\theta}^n$ and $\theta^1,\dots,\theta^n$ are related by $\bar{\theta}^i = \sum b^i_j \theta^j$, then with $B = [b^i_j]$,*

$$\bar{\theta}^1 \wedge \cdots \wedge \bar{\theta}^n = (\det B)\theta^1 \wedge \cdots \wedge \theta^n.$$

Proof. Problem 16.3. □

16.4 Volume Form

On an oriented manifold of dimension n there are many nowhere-vanishing n-forms. If ω is a nowhere-vanishing n-form, so is any nonzero multiple of ω. In general, it is not possible to single out one of them as the volume form. If the oriented manifold is Riemannian, however, we will show that there is a canonically defined volume form.

Let $(U, x^1,\dots,x^n)$ be an arbitrary chart on an oriented Riemannian manifold M. Applying the Gram–Schmidt process to the coordinate frame $\partial_1,\dots,\partial_n$, we can construct an orthonormal frame $e_1,\dots,e_n$ on U. Let $\theta^1,\dots,\theta^n$ be the dual frame of 1-forms and define

$$\omega = \theta^1 \wedge \cdots \wedge \theta^n.$$

Then ω is a nowhere-vanishing n-form on U.

To see how ω depends on the choice of an orthonormal frame, let $\bar{e}_1,\dots,\bar{e}_n$ be another orthonormal frame on U. Then there are C^∞ functions $a^i_j\colon U \to \mathbb{R}$ such that

$$\bar{e}_j = \sum a^i_j e_i. \tag{16.3}$$

Since both $e_1,\dots,e_n$ and $\bar{e}_1,\dots,\bar{e}_n$ are orthonormal, the change of basis matrix $A = [a^i_j]$ is a function from U to the orthogonal group $\mathrm{O}(n)$. Let $\bar{\theta}^1,\dots,\bar{\theta}^n$ be the dual frame of $\bar{e}_1,\dots,\bar{e}_n$. By Proposition 16.7, $\bar{\theta}^i = \sum (A^{-1})^i_j \theta^j$. It then follows from Proposition 16.8 that

$$\bar{\theta}^1 \wedge \cdots \wedge \bar{\theta}^n = \det(A^{-1})\theta^1 \wedge \cdots \wedge \theta^n.$$

We now assume that e and $\bar{e}$ are both positively oriented. Then the matrix function $A\colon U \to \mathrm{SO}(n)$ assumes values in the special orthogonal group. Since $A^{-1} \in \mathrm{SO}(n)$, $\det(A^{-1}) = 1$, and so

$$\bar{\theta}^1 \wedge \cdots \wedge \bar{\theta}^n = \theta^1 \wedge \cdots \wedge \theta^n.$$

This proves that $\omega = \theta^1 \wedge \cdots \wedge \theta^n$ is independent of the choice of the positively oriented orthonormal frame. It is a canonically defined n-form on an oriented Riemannian manifold M. We call ω the ***volume form*** of M. In case the integral $\int_M \omega$ is finite, we call $\int_M \omega$ the ***volume*** of the oriented Riemannian manifold M.

Example 16.9 (*Volume form on* $\mathbb{R}^2$). On $\mathbb{R}^2$, an orthonormal basis is $\partial/\partial x, \partial/\partial y$, with dual basis $\theta^1 = dx, \theta^2 = dy$. Hence, the volume form on $\mathbb{R}^2$ is $dx \wedge dy$.

Example 16.10 (*Volume form on* $\mathbb{H}^2$). By Section 12.5, an orthonormal frame on the Poincaré half-plane $\mathbb{H}^2$ is $e_1 = y\partial/\partial x$, $e_2 = y\partial/\partial y$, with dual frame $\theta^1 = (dx)/y$, $\theta^2 = (dy)/y$. Hence, the volume form on $\mathbb{H}^2$ is $dx \wedge dy/y^2$.

The notion of a tangent bundle extends to a manifold M with boundary ∂M (see [21, Section 22.4, pp. 253–254]); hence so does the notion of a Riemannian metric: a ***Riemannian metric*** on a manifold M with boundary is a positive-definite symmetric bilinear form on the tangent bundle TM. If M is a Riemannian manifold with boundary, then ∂M inherits naturally a Riemannian metric from M.

Theorem 16.11. *Let* vol_M *and* $\mathrm{vol}_{\partial M}$ *be the volume forms on an oriented Riemannian manifold* M *and on its boundary* ∂M*. Assume that the boundary is given the boundary orientation* [21, §22.6]. *If* X *is the outward unit normal along* ∂M*, then* $\mathrm{vol}_{\partial M} = \iota_X(\mathrm{vol}_M)$*, where* ι_X *is the interior multiplication with* X*.* (For the basic properties of interior multiplication, see [21, Section 20.4, pp. 227–229].)

Proof. Let p be a point in the boundary ∂M. If $(X, e_2,\dots,e_n)$ is a positively oriented orthonormal frame field for M in a neighborhood U of p, then by the definition of the boundary orientation, $(e_2,\dots,e_n)$ is a positively oriented orthonormal frame for ∂M in $U \cap \partial M$. Let $\theta^1,\dots,\theta^n$ be the dual frame to $X, e_2,\dots,e_n$. Clearly, $\theta^2,\dots,\theta^n$ is dual to $e_2,\dots,e_n$ on $U \cap \partial M$. So

$$\mathrm{vol}_M = \theta^1 \wedge \cdots \wedge \theta^n \quad \text{and} \quad \mathrm{vol}_{\partial M} = \theta^2 \wedge \cdots \wedge \theta^n.$$

At a point of ∂M,

$$\begin{aligned}\iota_X(\mathrm{vol}_M) &= \iota_X(\theta^1\wedge\cdots\wedge\theta^n)\\ &= \sum_j (-1)^{j-1}\theta^j(X)\ \theta^1\wedge\cdots\wedge\widehat{\theta^j}\wedge\cdots\wedge\theta^n\\ &= \theta^1(X)\ \theta^2\wedge\cdots\wedge\theta^n\\ &= \theta^2\wedge\cdots\wedge\theta^n = \mathrm{vol}_{\partial M}.\end{aligned}$$

□

Example 16.12 (*The volume form on a circle*). The volume form on the unit disk

$$D = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1\}$$

is $dx\wedge dy$. The unit outward normal X along its boundary $\partial D = S^1$ is

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

By Theorem 16.11, the volume form on the unit circle oriented counterclockwise is

$$\begin{aligned}\omega_{S^1} &= \iota_X(dx\wedge dy) = (\iota_X dx)dy - dx(\iota_X dy)\\ &= (Xx)\,dy - dx\,(Xy) = x\,dy - y\,dx.\end{aligned}$$

16.5 The Volume Form in Local Coordinates

On a coordinate chart $(U,x^1,\dots,x^n)$ of a Riemannian manifold M, the volume form vol_M, being a nowhere-vanishing top form, is a multiple $f\,dx^1\wedge\cdots\wedge dx^n$ of $dx^1\wedge\cdots\wedge dx^n$ for some nonvanishing function f on U. The next theorem determines f in terms of the Riemannian metric.

Theorem 16.13. *Let $(U,x^1,\dots,x^n)$ be a coordinate chart on a Riemannian manifold M, and let $g_{ij} = \langle\partial_i,\partial_j\rangle = \langle\partial/\partial x^i,\partial/\partial x^j\rangle$. Then the volume form of M on U is given by*

$$\mathrm{vol}_M = \sqrt{\det g}\,dx^1\wedge\cdots\wedge dx^n,$$

where g is the matrix $[g_{ij}]$.

Proof. Let $e_1,\dots,e_n$ be an orthonormal frame on U, with dual frame of 1-forms $\theta^1,\dots,\theta^n$. Then $\partial_j = \sum_i a^i_j e_i$ for some matrix-valued function $A = [a^i_j]\colon U\to \mathrm{GL}(n,\mathbb{R})$. The dual frame to $\partial_1,\dots,\partial_n$ is $dx^1,\dots,dx^n$. By Proposition 16.7, $dx^i = \sum_j b^i_j\theta^j$ with $B = [b^i_j] = A^{-1}$. By Proposition 16.8,

$$dx^1\wedge\cdots\wedge dx^n = (\det B)\theta^1\wedge\cdots\wedge\theta^n.$$

Hence,

$$\begin{aligned}\mathrm{vol}_M &= \theta^1\wedge\cdots\wedge\theta^n = (\det B)^{-1}dx^1\wedge\cdots\wedge dx^n\\ &= (\det A)\,dx^1\wedge\cdots\wedge dx^n.\end{aligned}$$

To find $\det A$, note that

$$g_{ij} = \langle \partial_i, \partial_j \rangle = \langle \sum a_i^k e_k, a_j^\ell e_\ell \rangle = \sum a_i^k a_j^\ell \langle e_k, e_\ell \rangle$$
$$= \sum a_i^k a_j^\ell \delta_{k\ell} = \sum_k a_i^k a_j^k = (A^T A)_j^i.$$

Therefore,

$$g = [g_{ij}] = A^T A,$$

so that

$$\det g = \det(A^T A) = (\det A)^2.$$

It follows that

$$\det A = \sqrt{\det g}$$

and

$$\mathrm{vol}_M = (\det A)\, dx^1 \wedge \cdots \wedge dx^n = \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n. \qquad \square$$

Problems

16.1. Transition matrix for orthonormal bases
Suppose $e_1, \ldots, e_n$ and $\bar{e}_1, \ldots, \bar{e}_n$ are two orthonormal bases in an inner product space V. Prove that if $\bar{e}_j = \sum a_j^i e_i$, then the matrix $A = [a_j^i]$ is orthogonal, i.e., $A^T A = I$.

16.2. The triangle inequality
Prove the triangle inequality for the distance function on a Riemannian manifold M: for all $p, q, r \in M$,

$$d(p,r) \le d(p,q) + d(q,r).$$

16.3. Wedge product under a change of frame
Prove Proposition 16.8.

16.4. Volume form of a sphere in Cartesian coordinates
Let $S^{n-1}(a)$ be the sphere of radius a centered at the origin in $\mathbb{R}^n$. Orient $S^{n-1}(a)$ as the boundary of the solid ball of radius a. Prove that if $x^1, \ldots, x^n$ are the Cartesian coordinates on $\mathbb{R}^n$, then the volume form on $S^{n-1}(a)$ is

$$\mathrm{vol}_{S^{n-1}(a)} = \frac{1}{a} \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

(*Hint*: Instead of finding an orthonormal frame on the sphere, use Theorem 16.11.)

16.5. Volume form of a 2-sphere in spherical coordinates
Parametrize the 2-sphere M of radius a in $\mathbb{R}^3$ by spherical coordinates θ, ϕ, where θ is the angle in the (x,y)-plane relative to the positive x-axis and ϕ is the angle relative to the positive z-axis (Figure 16.2). Then

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a\sin\phi\cos\theta \\ a\sin\phi\sin\theta \\ a\cos\phi \end{bmatrix}, \qquad 0 \le \theta < 2\pi, \quad 0 < \phi < \pi.$$

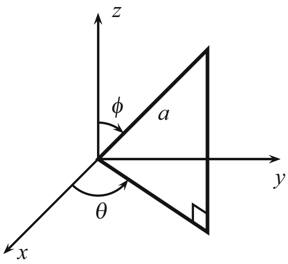


Fig. 16.2. Spherical coordinates in $\mathbb{R}^3$.

(a) Compute $\partial/\partial\phi$ and $\partial/\partial\theta$ and show that they are orthogonal to each other. (*Hint*: Use [21, Prop. 8.10, p. 90] to write $\partial/\partial\phi$ and $\partial/\partial\theta$ in terms of $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$.)
(b) Let $e_1 = (\partial/\partial\phi)/\|\partial/\partial\phi\|$, $e_2 = (\partial/\partial\theta)/\|\partial/\partial\theta\|$. Calculate the dual basis α^1, α^2 of 1-forms and the volume form on the 2-sphere of radius a in terms of a, ϕ, θ.
(c) Calculate the volume of M. (Since M is a surface, by its ***volume*** we mean its ***surface area***.)

16.6. Volume form of an n-sphere in spherical coordinates
For $n \geq 3$ we define the spherical coordinates on $\mathbb{R}^n$ as follows (see Figure 16.3). Let r_k be the distance of the point $(x_1,\dots,x_k)$ from the origin in $\mathbb{R}^k$:

$$r_k = \sqrt{x_1^2 + \cdots + x_k^2}.$$

(In this problem we use subscripts instead of superscripts on coordinate functions so that x_1^2 means the square of x_1.) The spherical coordinates on $\mathbb{R}^2$ are the usual polar coordinates $r = r_2$, θ, $0 \leq \theta < 2\pi$. For $n \geq 3$, if $x = (x_1,\dots,x_n)$, then the angle ϕ_n is the angle the vector x makes relative to the x_n-axis; it is determined uniquely by the formula

$$\cos\phi_n = \frac{x_n}{r_n}, \quad 0 \leq \phi_n \leq \pi.$$

Project x to $\mathbb{R}^{n-1}$ along the x_n-axis. By induction the spherical coordinates r_{n-1}, θ, ϕ_3, …, ϕ_{n-1} of the projection $(x_1,\dots,x_{n-1})$ in $\mathbb{R}^{n-1}$ are defined. Then the spherical coordinates of $(x_1,\dots,x_n)$ in $\mathbb{R}^n$ are defined to be $r_n, \theta, \phi_3, \dots, \phi_{n-1}, \phi_n$. Thus for $k = 3,\dots,n$, we have $\cos\phi_k = x_k/r_k$. Let $S^{n-1}(a)$ be the sphere $r_n = a$.

(a) Find the length of the vectors $\partial/\partial\theta$, $\partial/\partial\phi_k$ for $k = 3,\dots,n$.
(b) Normalize the vectors in part (a) to obtain an orthonormal frame $e_2,\dots,e_n$ on the sphere $S^{n-1}(a)$.
(c) Give the sphere $S^{n-1}(a)$ the boundary orientation of the closed solid ball of radius a. (For a discussion of boundary orientation, see [21, Section 21.6, p. 255].) Show that the volume form on $S^{n-1}(a)$ in spherical coordinates is up to sign

$$\omega = a^{n-1}(\sin^{n-2}\phi_n)(\sin^{n-3}\phi_{n-1})\cdots(\sin\phi_3)\,d\theta\wedge d\phi_3\wedge\cdots\wedge d\phi_n.$$

16.7. Surface area of a sphere
Because the "volume of a sphere" usually means the volume of the ball enclosed by the sphere, we will call the integral of the volume form on a sphere the ***surface area*** of the sphere.

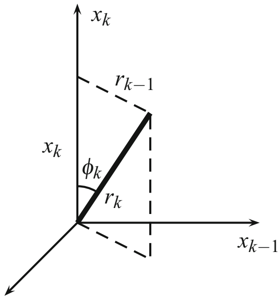


Fig. 16.3. Spherical coordinates in $\mathbb{R}^n$.

(a) By integration by parts, show that

$$\int_0^\pi \sin^n \phi \, d\phi = \frac{n-1}{n} \int_0^\pi \sin^{n-2} \phi \, d\phi.$$

(b) Give a numerical expression for $\int_0^\pi \sin^{2k} \phi \, d\phi$ and $\int_0^\pi \sin^{2k-1} \phi \, d\phi$.

(c) Compute the surface area of a sphere of radius a. Treat the even-dimensional case and the odd-dimensional case separately.

16.8. Volume form on a smooth hypersurface in $\mathbb{R}^n$

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function whose partial derivatives $\partial f/\partial x^i$ are not simultaneously zero at any point of the zero locus $Z(f)$ of f. By the regular level set theorem, $Z(f)$ is a smooth manifold of dimension $n-1$ (see [21, Th. 9.9, p. 105]). It inherits a Riemannian metric from $\mathbb{R}^n$.

(a) Show that the ***gradient vector field*** grad $f = \sum(\partial f/\partial x^i)\partial/\partial x^i$ is a nowhere-vanishing normal vector field along $Z(f)$.

(b) The unit normal vector field $X = \operatorname{grad} f/\|\operatorname{grad} f\|$ on $Z(f)$ defines an orientation on $Z(f)$: an orthonormal frame $(e_2,\dots,e_n)$ at $p \in Z(f)$ is said to be positive for $T_p(Z(f))$ if and only if $(X, e_2,\dots,e_n)$ is positive for $T_p(\mathbb{R}^n)$. Compute the volume form on $Z(f)$ with this orientation.

§17 The Gauss–Bonnet Theorem

According to an elementary theorem in Euclidean geometry, the three angles of a triangle in the plane add up to 180°. This simple fact has a beautiful generalization to any polygon on a Riemannian 2-manifold. The formula, called the Gauss–Bonnet formula for a polygon, relates the sum of the angles of a polygon to the geodesic curvature of the edges of the polygon and the Gaussian curvature of the surface. As a corollary, the angles of a geodesic triangle on a Riemannian 2-manifold can add up to more or less than 180°, depending on whether the Gaussian curvature is everywhere positive or everywhere negative.

The Gauss–Bonnet formula for a polygon leads to the Gauss–Bonnet theorem for an oriented Riemannian 2-manifold, according to which the total Gaussian curvature $\int_M K \operatorname{vol}$ of an oriented Riemannian 2-manifold M is equal to 2π times the Euler characteristic $\chi(M)$. The most striking feature of this theorem is that while the Gaussian curvature K depends on the Riemannian metric, the Euler characteristic does not. Thus, although the Gaussian curvature depends on the metric, its integral does not. Read another way, the Gauss–Bonnet theorem also implies that although the Euler characteristic by definition depends on a polygonal decomposition of the surface, because the total curvature does not, the Euler characteristic is in fact independent of the polygonal decomposition.

It follows that two homeomorphic compact orientable surfaces have the same Euler characteristic, since a homeomorphism carries a polygonal decomposition from one surface to the other and preserves the number of vertices, edges, and faces. Thus, at least for compact orientable surfaces, the Euler characteristic is a topological invariant. It is the first instance of a topological invariant that can be constructed using curvature.

17.1 Geodesic Curvature

In the same way the curvature of a curve in Euclidean space is defined using the usual derivative (Problem 2.7), we can define the ***geodesic curvature*** of a curve in a Riemannian manifold using the covariant derivative along the curve.

Consider a unit-speed curve $\gamma(s)\colon [a,b] \to M$ in a Riemannian manifold M. The Riemannian connection of M induces a covariant derivative D/ds along the curve. Let $T(s) = \gamma'(s)$. The curve $\gamma(s)$ is a geodesic if and only if DT/ds vanishes identically. Thus, the magnitude $\|DT/ds\|$ gives a measure of the extent to which $\gamma(s)$ fails to be a geodesic. It is called the ***geodesic curvature*** $\tilde{\kappa}_g$ of the curve $\gamma(s)$:

$$\tilde{\kappa}_g = \left\| \frac{DT}{ds} \right\|.$$

So defined, the geodesic curvature is always nonnegative.

It follows directly from the definition that a unit-speed curve is a geodesic if and only if its geodesic curvature is zero.

17.2 The Angle Function Along a Curve

At a point p of an oriented Riemannian manifold M, the angle ζ between two vectors u and v in the tangent space T_pM is given by the formula

$$\cos\zeta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|}.$$

In general, the angle ζ is defined only up to an integer multiple of 2π.

Suppose now that (U, e_1, e_2) is a framed open set on which there is a positively oriented orthonormal frame e_1, e_2. If $c\colon [a,b] \to U$ is a C^∞ curve, let $\underline{\zeta}(t)$ be the angle from $e_{1,c(t)}$ to $c'(t)$ in $T_{c(t)}M$. Because $\underline{\zeta}(t)$ is defined only up to an integer multiple of 2π, $\underline{\zeta}$ is a function from $[a,b]$ to $\mathbb{R}/2\pi\mathbb{Z}$. It is a C^∞ function, since it is locally a branch of $\cos^{-1}\langle c'(t), e_{1,c(t)}\rangle / \|c'(t)\|$. Let ζ_0 be a real number such that

$$\cos\zeta_0 = \frac{\langle c'(a), e_{1,c(a)}\rangle}{\|c'(a)\|}.$$

Since $\mathbb{R}$ is a covering space of $\mathbb{R}/2\pi\mathbb{Z}$ and the interval $[a,b]$ is simply connected, we know from the theory of covering spaces that there is a unique C^∞ map $\zeta\colon [a,b] \to \mathbb{R}$ with a specified initial value $\zeta(a) = \zeta_0$ that covers $\underline{\zeta}$:

$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{\zeta}{\nearrow} & \downarrow \\ [a,b] & \xrightarrow[\underline{\zeta}]{} & \mathbb{R}/2\pi\mathbb{Z}. \end{array}$$

We call $\zeta\colon [a,b] \to \mathbb{R}$ the ***angle function*** with initial value ζ_0 along the curve c relative to the frame e_1, e_2. Since an angle function along a curve is uniquely determined by its initial value, any two angle functions along the curve c relative to e_1, e_2 differ by a constant integer multiple of 2π.

17.3 Signed Geodesic Curvature on an Oriented Surface

Mimicking the definition of the signed curvature for a plane curve, we can give the geodesic curvature a sign on an oriented Riemannian 2-manifold. Given a unit-speed curve $\gamma(s)\colon [a,b] \to M$ on an oriented Riemannian 2-manifold M, choose a unit vector field $\mathbf{n}$ along the curve so that $T, \mathbf{n}$ is positively oriented and orthonormal. Since $\langle T, T\rangle \equiv 1$ on the curve,

$$\frac{d}{ds}\langle T, T\rangle = 0.$$

By Theorem 13.2,

$$2\left\langle \frac{DT}{ds}, T \right\rangle = 0.$$

Since the tangent space of M at $\gamma(s)$ is 2-dimensional and DT/ds is perpendicular to T, DT/ds must be a scalar multiple of $\mathbf{n}$.

Definition 17.1. The ***signed geodesic curvature*** at a point $\gamma(s)$ of a unit-speed curve in an oriented Riemannian 2-manifold is the number $\kappa_g(s)$ such that

$$\frac{DT}{ds} = \kappa_g \mathbf{n}.$$

The signed geodesic curvature κ_g can also be computed as

$$\kappa_g = \left\langle \frac{DT}{ds}, \mathbf{n} \right\rangle.$$

Let U be an open subset of the oriented Riemannian 2-manifold M such that there is an orthonormal frame e_1, e_2 on U. We assume that the frame e_1, e_2 is positively oriented. Suppose $\gamma\colon [a,b] \to U$ is a unit-speed curve. Let $\zeta\colon [a,b] \to U$ be an angle function along the curve γ relative to e_1, e_2. Thus, $\zeta(s)$ is the angle that the velocity vector $T(s)$ makes relative to e_1 at $\gamma(s)$ (Figure 17.1). In terms of the angle ζ,

$$T = \quad (\cos\zeta)e_1 + (\sin\zeta)e_2, \tag{17.1}$$

$$\mathbf{n} = -(\sin\zeta)e_1 + (\cos\zeta)e_2, \tag{17.2}$$

where e_1, e_2 are evaluated at $c(t)$.

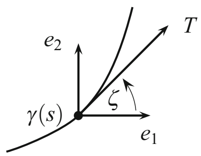


Fig. 17.1. The angle ζ.

Proposition 17.2. *Let ω_2^1 be the connection form of an affine connection on a Riemannian 2-manifold relative to the positively oriented orthonormal frame e_1, e_2 on U. Then the signed geodesic curvature of the unit-speed curve γ is given by*

$$\kappa_g = \frac{d\zeta}{ds} - \omega_2^1(T).$$

Proof. Differentiating (17.1) with respect to s gives

$$\frac{DT}{ds} = \left(\frac{d}{ds}\cos\zeta\right)e_1 + (\cos\zeta)\frac{De_1}{ds} + \left(\frac{d}{ds}\sin\zeta\right)e_2 + (\sin\zeta)\frac{De_2}{ds}.$$

In this sum, e_i really means $e_{i,c(t)}$ and $De_{i,c(t)}/ds = \nabla_T e_i$ by Theorem 13.1(iii), so that by Proposition 11.4,

$$\frac{De_1}{ds} = \nabla_T e_1 = \omega_1^2(T)e_2 = -\omega_2^1(T)e_2,$$
$$\frac{De_2}{ds} = \nabla_T e_2 = \omega_2^1(T)e_1.$$

Hence,

$$\begin{aligned}\frac{DT}{ds} &= -(\sin\zeta)\frac{d\zeta}{ds}e_1 - (\cos\zeta)\omega_2^1(T)e_2 \\ &\quad + (\cos\zeta)\frac{d\zeta}{ds}e_2 + (\sin\zeta)\omega_2^1(T)e_1 \\ &= \left(\frac{d\zeta}{ds} - \omega_2^1(T)\right)\mathbf{n}.\end{aligned}$$

So

$$\kappa_g = \frac{d\zeta}{ds} - \omega_2^1(T). \qquad \square$$

Since κ_g is a C^∞ function on the interval $[a,b]$, it can be integrated. The integral $\int_a^b \kappa_g\,ds$ is called the ***total geodesic curvature*** of the unit-speed curve $\gamma\colon [a,b] \to M$.

Corollary 17.3. *Let M be an oriented Riemannian 2-manifold. Assume that the image of the unit-speed curve $\gamma\colon [a,b] \to M$ is a 1-dimensional submanifold C with boundary. If C lies in an open set U with positively oriented orthonormal frame e_1, e_2 and connection form ω_2^1, then its total geodesic curvature is*

$$\int_a^b \kappa_g\,ds = \zeta(b) - \zeta(a) - \int_C \omega_2^1.$$

Proof. Note that $\gamma^{-1} : C \to [a,b]$ is a coordinate map on C, so that

$$\int_C \omega_2^1 = \int_a^b \gamma^*\omega_2^1.$$

Let s be the coordinate on $[a,b]$. Then

$$\gamma^*\omega_2^1 = f(s)ds \tag{17.3}$$

for some C^∞ function $f(s)$. To find $f(s)$, apply both sides of (17.3) to d/ds:

$$f(s) = (\gamma^*\omega_2^1)\left(\frac{d}{ds}\right) = \omega_2^1\left(\gamma_*\frac{d}{ds}\right) = \omega_2^1(\gamma'(s)) = \omega_2^1(T).$$

Hence, $\gamma^* \omega_2^1 = \omega_2^1(T)ds$. By Proposition 17.2,

$$\begin{aligned}\int_a^b \kappa_g\, ds &= \int_a^b \frac{d\zeta}{ds}\, ds - \int_a^b \omega_2^1(T)\, ds \\ &= \zeta(b) - \zeta(a) - \int_a^b \omega_2^1(T)\, ds \\ &= \zeta(b) - \zeta(a) - \int_a^b \gamma^* \omega_2^1 \\ &= \zeta(b) - \zeta(a) - \int_C \omega_2^1. \end{aligned}$$

□

17.4 Gauss–Bonnet Formula for a Polygon

A ***polygon*** on a surface M is a piecewise smooth simple closed curve $\gamma\colon [a,b] \to M$; here "closed" means that $\gamma(a) = \gamma(b)$ and "simple" means that the curve has no other self-intersections. We say that a polygon is ***unit-speed*** if it has unit speed everywhere except at the nonsmooth points. Suppose now that M is an oriented Riemannian 2-manifold and that $\gamma\colon [a,b] \to M$ is a unit-speed polygon that lies in an open set U on which there is an oriented orthonormal frame e_1, e_2. Let $\gamma(s_0), \gamma(s_1), \dots, \gamma(s_m)$ be the vertices of the curve, where $a = s_0, b = s_m$, and $\gamma(a) = \gamma(b)$. Denote by C the image of γ, and by R the region enclosed by the curve C (Figure 17.2). Let K be the Gaussian curvature of the oriented Riemannian 2-manifold M.

In a neighborhood of a vertex $\gamma(s_i)$, the curve is the union of two 1-dimensional manifolds with boundary. Let $\gamma'(s_i^-)$ be the outward-pointing tangent vector at $\gamma(s_i)$ of the incoming curve, and let $\gamma'(s_i^+)$ be the inward-pointing tangent vector at $\gamma(s_i)$ of the outgoing curve. The angle ε_i in $]-\pi, \pi[$ from the incoming tangent vector $\gamma'(s_i^-)$ to the outgoing tangent vector $\gamma'(s_i^+)$ is called the ***jump angle*** at $\gamma(s_i)$. The ***interior angle*** at $\gamma(s_i)$ is defined to be

$$\beta_i := \pi - \varepsilon_i.$$

Along each edge $\gamma([s_{i-1}, s_i])$ of a polygon, the velocity vector T changes its angle by $\Delta\zeta_i = \zeta(s_i) - \zeta(s_{i-1})$. At the vertex $\gamma(s_i)$, the angle jumps by ε_i. Thus, the total change in the angle of T around a polygon is

$$\sum_{i=1}^m \Delta\zeta_i + \sum_{i=1}^m \varepsilon_i.$$

In Figure 17.2, it appears that the total change in the angle is 2π. This is the content of the Hopf Umlaufsatz (circulation theorem), which we state below. On an oriented surface, a polygon is ***positively oriented*** if it has the boundary orientation of the oriented region it encloses. In the plane, positive orientation is counterclockwise.

Theorem 17.4 (Hopf Umlaufsatz). *Let (U, e_1, e_2) be a framed open set on an oriented Riemannian 2-manifold, and $\gamma\colon [a,b] \to U$ a positively oriented piecewise smooth simple closed curve. Then the total change in the angle of $T(s) = \gamma'(s)$ around γ is*

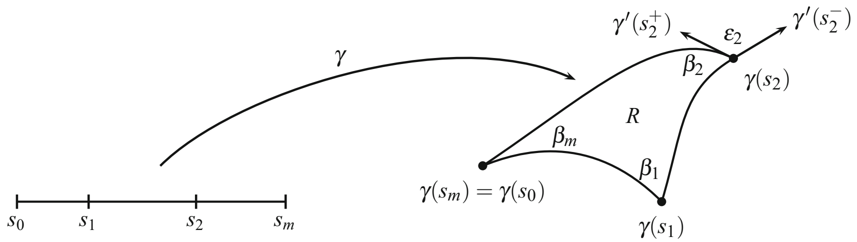


Fig. 17.2. A polygonal region on a surface.

$$\sum_{i=1}^{m} \Delta\zeta_i + \sum_{i=1}^{m} \varepsilon_i = 2\pi.$$

The Hopf Umlaufsatz is also called the rotation index theorem or the rotation angle theorem. A proof may be found in [11] or [16, Th. 2.9, pp. 56–57]. In the Addendum we give a proof for the case of a plane curve.

Theorem 17.5 (Gauss–Bonnet formula for a polygon). *Under the hypotheses above,*

$$\int_a^b \kappa_g\, ds = 2\pi - \sum_{i=1}^{m} \varepsilon_i - \int_R K \operatorname{vol}.$$

Proof. The integral $\int_a^b \kappa_g ds$ is the sum of the total geodesic curvature on each edge of the simple closed curve. As before, we denote by $\Delta\zeta_i$ the change in the angle ζ along the ith edge of γ. By Corollary 17.3,

$$\begin{aligned} \int_a^b \kappa_g\, ds &= \sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} \kappa_g\, ds \\ &= \sum_i \Delta\zeta_i - \int_C \omega_2^1. \end{aligned} \tag{17.4}$$

By the Hopf Umlaufsatz,

$$\sum \Delta\zeta_i = 2\pi - \sum \varepsilon_i. \tag{17.5}$$

Recall from Section 12 that on a framed open set (U, e_1, e_2) of a Riemannian 2-manifold with orthonormal frame e_1, e_2 and dual frame θ^1, θ^2, if $[\Omega_j^i]$ is the curvature matrix relative to the frame e_1, e_2, then

$$\Omega_2^1 = d\omega_2^1 = K\,\theta^1 \wedge \theta^2.$$

Therefore,

$$\begin{aligned} \int_C \omega_2^1 &= \int_{\partial R} \omega_2^1 \\ &= \int_R d\omega_2^1 \qquad \text{(by Stokes' theorem)} \\ &= \int_R K\theta^1 \wedge \theta^2 = \int_R K \operatorname{vol}. \end{aligned} \tag{17.6}$$

Combining (17.4), (17.5), and (17.6) gives the Gauss–Bonnet formula for a polygon. □

17.5 Triangles on a Riemannian 2-Manifold

A ***geodesic polygon*** is a polygon whose segments are geodesics. We will deduce from the Gauss–Bonnet formula a famous result on the sum of the angles of a geodesic triangle on a Riemannian 2-manifold.

Proposition 17.6. *Suppose a geodesic m-gon γ lies in a framed open set of an oriented Riemannian 2-manifold. If β_i is the interior angle at vertex i of the geodesic m-gon, R is the closed region enclosed by γ, and K is the Gaussian curvature on R, then*

$$\sum_{i=1}^{m} \beta_i = (m-2)\pi + \int_R K \operatorname{vol}.$$

Proof. Along a geodesic m-gon, the geodesic curvature κ_g is identically zero. The exterior angle ε_i at vertex i is

$$\varepsilon_i = \pi - \beta_i.$$

Hence, the Gauss–Bonnet formula becomes

$$0 = 2\pi - \sum_{i=1}^{m} (\pi - \beta_i) - \int_R K \operatorname{vol},$$

or

$$\sum_{i=1}^{m} \beta_i = (m-2)\pi + \int_R K \operatorname{vol}. \qquad \square$$

The following corollary is immediate.

Corollary 17.7. *Suppose a geodesic triangle lies in a framed open set of an oriented Riemannian 2-manifold M.*

(i) *If M has zero Gaussian curvature, the angles of the geodesic triangle add up to π radians.*
(ii) *If M has everywhere positive Gaussian curvature, the angles of the geodesic triangle add up to more than π radians.*
(iii) *If M has everywhere negative Gaussian curvature, the angles of the geodesic triangle add up to less than π radians.*

By Section 12.5, the Poincaré half-plane has constant Gaussian curvature -1. It is also known as the ***hyperbolic plane***. We will call a geodesic triangle in the Poincaré half-plane a ***hyperbolic triangle***.

Corollary 17.8. *The area enclosed by a hyperbolic triangle is π minus the sum of its interior angles.*

Proof. If R is the region enclosed by the hyperbolic triangle, then by Proposition 17.6,

$$\begin{aligned} \operatorname{Area}(R) &= \int_R 1 \operatorname{vol} = -\int_R K \operatorname{vol} \\ &= \pi - \sum_{i=1}^{3} \beta_i. \end{aligned} \qquad \square$$

17.6 Gauss–Bonnet Theorem for a Surface

Let M be a compact oriented Riemannian 2-manifold. Suppose M has been cut up into polygonal regions, each of which is bounded by a piecewise smooth simple closed curve and lies in a framed open set. We call such a polygonal decomposition ***sufficiently fine***. Let V, E, F be the number of vertices, edges, and faces, respectively, of this polygonal decomposition. The ***Euler characteristic*** of the decomposition is defined to be

$$\chi = V - E + F.$$

Theorem 17.9 (Gauss–Bonnet theorem). *For a compact oriented Riemannian* 2-*manifold M,*

$$\int_M K\,\mathrm{vol} = 2\pi\chi(M).$$

Proof. Decompose the surface M into a sufficiently fine polygonal decomposition. The orientation on the surface will orient each face in such a way that each edge is oriented in two opposite ways, depending on which of the two adjacent faces it is a boundary of (see Figure 17.3).

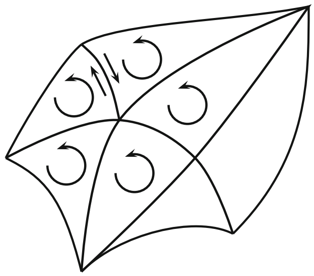


Fig. 17.3. Orienting a triangulated surface.

Denote the polygonal regions by R_j. If we sum up $\int \kappa_g\,ds$ over the boundaries of all the regions R_j, the result is zero because each edge occurs twice, with opposite orientations. Let $\beta_i^{(j)}$, $1 \le i \le n_j$, be the interior angles of the jth polygon. Then by the Gauss–Bonnet formula for a polygon (Theorem 17.5),

$$0 = \sum_j \int_{\partial R_j} \kappa_g\,ds = 2\pi F - \sum_j \sum_{i=1}^{n_j} (\pi - \beta_i^{(j)}) - \sum_j \int_{R_j} K\,\mathrm{vol}.$$

In the sum $\sum_j \sum_i (\pi - \beta_i^{(j)})$, the term π occurs $2E$ times because each edge occurs twice. The sum of all the interior angles is 2π at each vertex; hence,

$$\sum_{i,j} \beta_i^{(j)} = 2\pi V.$$

Therefore,

$$0 = 2\pi F - 2\pi E + \sum_{i,j} \beta_i^{(j)} - \int_M K \,\mathrm{vol},$$

or

$$\begin{aligned}\int_M K \,\mathrm{vol} &= 2\pi V - 2\pi E + 2\pi F \\ &= 2\pi\chi(M).\end{aligned}$$

□

In the Gauss–Bonnet theorem the total curvature $\int_M K \,\mathrm{vol}$ on the left-hand side is independent of the polygonal decomposition while the Euler characteristic $2\pi\chi$ on the right-hand side is independent of the Riemannian structure. This theorem shows that the Euler characteristic of a sufficiently fine polygonal decomposition is independent of the decomposition and that the total curvature is a topological invariant, independent of the Riemannian structure.

By the classification theorem for surfaces, a compact orientable surface is classified by an integer g called its *genus*, which is essentially the number of holes in the surface (Figure 17.4).

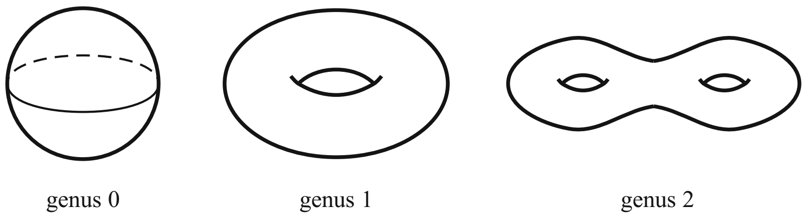


Fig. 17.4. Compact orientable surfaces.

Any two compact orientable surfaces of the same genus are diffeomorphic and moreover, if M is a compact orientable surface of genus g, then

$$\chi(M) = 2 - 2g.$$

Theorem 17.10. *If a compact orientable Riemannian manifold of dimension* 2 *has positive Gaussian curvature K everywhere, then it is diffeomorphic to a sphere.*

Proof. By the Gauss–Bonnet theorem,

$$2\pi\chi(M) = \int_M K \,\mathrm{vol} > 0.$$

So $\chi(M) = 2 - 2g > 0$, which is equivalent to $g < 1$. Hence, $g = 0$ and M is diffeomorphic to a sphere. □

17.7 Gauss–Bonnet Theorem for a Hypersurface in $\mathbb{R}^{2n+1}$

If M is an even-dimensional smooth, compact, oriented hypersurface in $\mathbb{R}^{2n+1}$, we defined its Gaussian curvature K in Section 9.1. The Gauss–Bonnet theorem for a hypersurface states that

$$\int_M K\mathrm{vol}_M = \frac{\mathrm{vol}(S^{2n})}{2}\chi(M).$$

For a proof, see [19, Vol. 5, Ch. 13, Th. 26, p. 442]. For $n = 1$, $\mathrm{vol}(S^2) = 4\pi$ and this theorem agrees with Theorem 17.9.

Problems

17.1. Interior angles of a polygon
Use the Gauss–Bonnet formula to find the sum of the interior angles of a geodesic m-gon in the Euclidean plane.

17.2. Total curvature of a plane curve
Let C be a smooth, simple closed plane curve parametrized counterclockwise by $\gamma(s)$, where $s \in [0,\ell]$ is the arc length. If $k(s)$ is the signed curvature of the curve at $\gamma(s)$, prove that

$$\int_0^\ell k(s)\,ds = 2\pi.$$

This is a restatement of Hopf's Umlaufsatz, since if θ is the angle of the tangent vector $\gamma'(s)$ relative to the positive x-axis, then

$$\int_0^\ell k(s)\,ds = \int_0^\ell \frac{d\theta}{ds}\,ds = \theta(\ell) - \theta(0),$$

is the change in the angle θ from $s = 0$ to $s = \ell$.

17.3. The Gauss map and volume
Let M be a smooth, compact, oriented surface in $\mathbb{R}^3$. The Gauss map $\nu\colon M \to S^2$ of M is defined in Problem 5.3. If vol_M and vol_{S^2} are the volume forms on M and on S^2 respectively, prove that

$$\nu^*(\mathrm{vol}_{S^2}) = K\,\mathrm{vol}_M,$$

where K is the Gaussian curvature on M.

17.4. Degree of the Gauss map of a surface
If $f\colon M \to N$ is a map of compact oriented manifolds of the same dimension, the ***degree*** of f is defined to be the number

$$\deg f = \int_M f^*\omega$$

where ω is a top form on N with $\int_N \omega = 1$ [3, p. 40]. Prove that for a smooth, compact, oriented surface M in $\mathbb{R}^3$, the degree of the Gauss map $\nu\colon M \to S^2$ is

$$\deg \nu = \frac{1}{2}\chi(M).$$

(*Hint*: Take ω to be $\mathrm{vol}_{S^2}/4\pi$. Then $\deg\nu = \int_M \nu^*(\mathrm{vol}_{S^2})/4\pi$. Use Problem 17.3 and the Gauss–Bonnet theorem.)

17.5. Degree of the Gauss map of a hypersurface
Let M be a smooth, compact, oriented $2n$-dimensional hypersurface in $\mathbb{R}^{2n+1}$. Let N be the unit outward normal vector field on M. The ***Gauss map*** $\nu\colon M \to S^{2n}$ assigns to each point $p \in M$ the unit outward normal N_p. Using the Gauss–Bonnet theorem for a hypersurface, prove that the degree of the Gauss map is

$$\deg \nu = \frac{1}{2}\chi(M).$$

Addendum. Hopf Umlaufsatz in the Plane

In this Addendum we give a proof of the Hop Umlaufsatz in the plane, more or less along the line of Hopf's 1935 paper [11].

Let $c\colon [a,b] \to \mathbb{R}^2$ be a regular curve (Recall that "regular" means that c' is never zero). Let $\bar{\zeta}\colon [a,b] \to \mathbb{R}/2\pi\mathbb{Z}$ be the angle that the unit tangent vector makes with respect to the positive x-axis. Because angles are defined up to an integral multiple of 2π, it is a C^∞ function with values in $\mathbb{R}/2\pi\mathbb{Z}$. Since the interval $[a,b]$ is simply connected, $\bar{\zeta}$ has a C^∞ lift $\zeta\colon [a,b] \to \mathbb{R}$, which is unique if we fix the initial value $\zeta(a) \in \mathbb{R}$. If c is piecewise smooth with vertices at $s_0 < s_1 < \cdots < s_r$, then at each vertex $c(s_i)$, the angle at the endpoint of one segment and the jump angle there determines uniquely the angle at the initial point of the next segment. Thus, on a piecewise smooth curve there is a well-defined C^∞ angle function on each segment once the initial angle $\zeta(a) \in \mathbb{R}$ is specified.

Theorem 17.1 (Hopf Umlaufsatz for a plane curve). *Let $\gamma\colon [0,\ell] \to \mathbb{R}^2$ be a piecewise smooth simple closed curve oriented counterclockwise and parametrized by arc length. Then the change in the angle around γ is*

$$\zeta(\ell) - \zeta(0) = 2\pi.$$

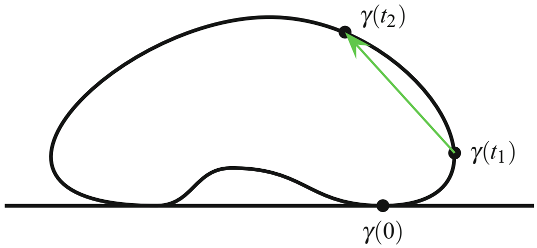


Fig. 17.5. A simple closed curve.

Proof. Let $C = \gamma([0,\ell])$ be the image of the parametrized curve. First consider the case when C is smooth. Since C is compact, it is bounded and therefore lies above some horizontal line. Parallel translate this horizontal line upward until it is tangent to C. Translate the coordinate system so that the origin $O = (0,0)$ is a point of tangency with a horizontal tangent (Figure 17.5).

For any two points $\gamma(t_1), \gamma(t_2)$ on C with $t_1 < t_2$, let $\bar{\zeta}(t_1,t_2)$ be the angle that the unit secant vector $\overrightarrow{\gamma(t_1)\gamma(t_2)}/\|\overrightarrow{\gamma(t_1)\gamma(t_2)}\|$ makes relative to the positive x-axis. If $t_1 = t_2$, let $\bar{\zeta}(t_1,t_2)$ be the angle that the unit tangent vector at $\gamma(t_1)$ makes relative to the positive x-axis.

Let T be the closed triangle in $\mathbb{R}^2$ with vertices $O = (0,0)$, $A = (0,\ell)$, and $L = (\ell,\ell)$ (Figure 17.6). The angle function $\bar{\zeta}\colon T \to \mathbb{R}/2\pi\mathbb{Z}$ is continuous on the closed rectangle T and C^∞ on the open rectangle. Since T is simply connected, $\bar{\zeta}$ lifts to a continuous function $\zeta\colon T \to \mathbb{R}$ that is C^∞ in the interior of T.

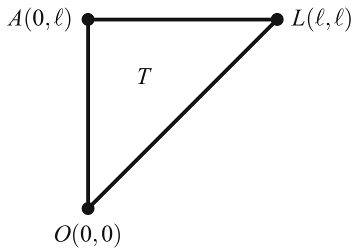


Fig. 17.6. Domain of the secant angle function.

Then $d\zeta$ is a well-defined 1-form on T, and the change in the angle ζ around the curve γ is given by the integral

$$\int_{\overline{OL}} d\zeta = \zeta(L) - \zeta(O) = \zeta(\ell) - \zeta(0).$$

By Stokes' theorem,

$$\int_{\partial T} d\zeta = \int_T d(d\zeta) = 0,$$

where $\partial T = \overline{OL} + \overline{LA} + \overline{AO} = \overline{OL} - \overline{AL} - \overline{OA}$. Hence,

$$\int_{\overline{OL}} d\zeta = \int_{\overline{OA}} d\zeta + \int_{\overline{AL}} d\zeta. \tag{17.7}$$

On the right the first integral $\int_{\overline{OA}} d\zeta = \zeta(0,\ell) - \zeta(0,0)$ is the change in the angle as the secant moves from $(0,0)$ to $(0,\ell)$, i.e., the initial point of the secant is fixed at O while the endpoint moves from O to O counterclokwise along the curve (Figure 17.7). Thus,

$$\int_{\overline{OA}} d\zeta = \zeta(0,\ell) - \zeta(0,0) = \pi - 0 = \pi.$$

The second integral on the right in (17.7),

$$\int_{\overline{AL}} d\zeta = \zeta(\ell,\ell) - \zeta(0,\ell),$$

is the change in the angle of the secant if the initial point of the secant moves from O to O counterclockwise along the curve while the endpoint is fixed at O (Figure 17.8). Thus,

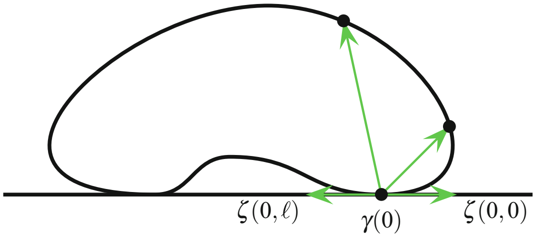


Fig. 17.7. Change in the secant angle from $(0,0)$ to $(0,\ell)$.

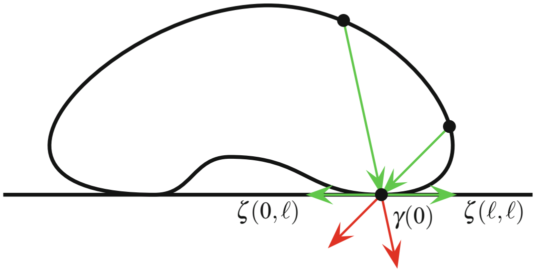


Fig. 17.8. Change in the secant angle from $(0,\ell)$ to (ℓ,ℓ).

$$\int_{\overline{AL}} d\zeta = \zeta(\ell,\ell) - \zeta(0,\ell) = 2\pi - \pi = \pi.$$

So in (17.7),

$$\zeta(\ell) - \zeta(0) = \int_{\overline{OL}} d\zeta = \int_{\overline{OA}} d\zeta + \int_{\overline{AL}} d\zeta = \pi + \pi = 2\pi.$$

This proves the Hopf Umlaufsatz for a smooth curve γ.

If $\gamma\colon [0,\ell] \to \mathbb{R}^2$ is piecewise smooth so that C has corners, we can "smooth a corner" by replacing it by a smooth arc and thereby removing the singularity (Figure 17.9).

In a neighborhood of each corner, the change in the angle is the change in the angle over the smooth pieces plus the jump angle at the corner. It is the same as the change in the angle of the smoothed curve. Hence,

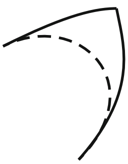


Fig. 17.9. Smoothing a corner.

$$\zeta(\ell) - \zeta(0) = \sum \Delta\zeta_i + \sum \varepsilon_i = 2\pi.$$

Chapter 4

Tools from Algebra and Topology

This chapter is a digression in algebra and topology. We saw earlier that the tangent space construction gives rise to a functor from the category of C^∞ manifolds (M, p) with a marked point to the category of vector spaces. Much of differential topology and differential geometry consists of trying to see how much of the geometric information is encoded in the linear algebra of tangent spaces. To this end, we need a larger arsenal of algebraic techniques than linear spaces and linear maps.

In Sections 18 and 19, we introduce the tensor product, the dual, the Hom functor, and the exterior power. These functors are applied in Section 20 to the fibers of vector bundles in order to form new vector bundles. In Section 21, we generalize differential forms with values in $\mathbb{R}$ to differential forms with values in a vector space or even in a vector bundle. The curvature form of a connection on a vector bundle E is an example of a 2-form with values in a vector bundle, namely, the endomorphism bundle $\mathrm{End}(E)$.

Although we are primarily interested in the tensor product of vector spaces, it is not any more difficult to define the tensor product of modules. This could come in handy in algebraic topology, where one might want to tensor two abelian groups, or in the theory of vector bundles, where the space of C^∞ sections of a bundle over a manifold is a module over the ring of C^∞ functions on the manifold. All the modules in this book will be left modules over a commutative ring R with identity. When the ring R is a field, the R-modules are vector spaces. The vector spaces may be infinite-dimensional, except where explicitly stated otherwise.

§18 The Tensor Product and the Dual Module

Some of the basic operations on a vector space are not linear. For example, any inner product on a vector space V is by definition bilinear, hence not linear on $V \times V$. In general, linear maps are easier to deal with than multilinear maps. The tensor product is a way of converting multilinear maps to linear maps. The main theorem

L.W. Tu, *Differential Geometry*, Graduate Texts in Mathematics 275,
DOI 10.1007/978-3-319-55084-8_4

(Theorem 18.3) states that bilinear maps on a Cartesian product $V \times W$ of vector spaces V and W correspond in a one-to-one manner to linear maps on the tensor product $V \otimes W$.

18.1 Construction of the Tensor Product

Let R be a commutative ring with identity. A subset B of a left R-module V is called a ***basis*** if every element of V can be written uniquely as a finite linear combination $\sum r_i b_i$, where $r_i \in R$ and $b_i \in B$. An R-module is said to be ***free*** if it has a basis, and if the basis is finite with n elements, then the free R-module is said to be ***of rank*** n.

Example 18.1. The $\mathbb{Z}$-module $\mathbb{Z}/2$ has no basis, for the only possible candidate for a basis element is 1, yet $0 = 0 \cdot 1 = 2 \cdot 1$, which shows that 0 is not uniquely a linear combination of 1.

Let V and W be left R-modules. To define their tensor product, first construct the free R-module $\mathrm{Free}(V \times W)$ whose basis is the set of all ordered pairs $(v,w) \in V \times W$. This means an element of $\mathrm{Free}(V \times W)$ is uniquely a ***finite*** linear combination of elements of $V \times W$:

$$\sum r_i(v_i, w_i), \quad r_i \in R, \quad (v_i, w_i) \in V \times W.$$

In $\mathrm{Free}(V \times W)$, consider the R-submodule S spanned by all elements of the form:

$$\begin{aligned} &(v_1 + v_2, w) - (v_1, w) - (v_2, w), \\ &(v, w_1 + w_2) - (v, w_1) - (v, w_2), \\ &(rv, w) - r(v, w), \\ &(v, rw) - r(v, w) \end{aligned} \tag{18.1}$$

for all $v_1, v_2, v \in V$, $w_1, w_2, w \in W$, and $r \in R$.

Definition 18.2. The ***tensor product*** $V \otimes_R W$ of two R-modules V and W is the quotient R-module $\mathrm{Free}(V \times W)/S$, where S is the R-submodule spanned by elements of the form (18.1). When it is understood that the coefficient ring is R, we write simply $V \otimes W$.

We denote the equivalence class of (v,w) by $v \otimes w$ and call it the ***tensor product*** of v and w. By construction, $v \otimes w$ is bilinear in its arguments:

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \tag{18.2}$$

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2, \tag{18.3}$$

$$(rv) \otimes w = r(v \otimes w) = v \otimes rw. \tag{18.4}$$

In other words, the canonical map

$$\begin{aligned} \otimes \colon V \times W &\to V \otimes W \\ (v, w) &\mapsto v \otimes w \end{aligned}$$

is bilinear.

Elements of the form $v \otimes w$ are said to be ***decomposable*** in $V \otimes W$. By construction every element in $V \otimes W$ is a finite sum of decomposable elements:

$$\sum r_i(v_i \otimes w_i) = \sum (r_i v_i) \otimes w_i.$$

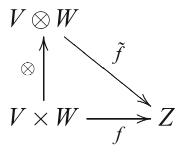
The decomposition into a finite sum of decomposable elements is not unique; for example,

$$v_1 \otimes w + v_2 \otimes w = (v_1 + v_2) \otimes w.$$

18.2 Universal Mapping Property for Bilinear Maps

The tensor product satisfies the following universal mapping property.

Theorem 18.3. *Let V, W, Z be left modules over a commutative ring R with identity. Given any R-bilinear map $f\colon V \times W \to Z$, there is a unique R-linear map $\tilde{f}\colon V \otimes W \to Z$ such that the diagram*



(18.5)

commutes.

The commutativity of the diagram means that $\tilde{f} \circ \otimes = f$, or

$$\tilde{f}(v \otimes w) = f(v, w) \quad \text{for all } (v, w) \in V \times W.$$

Proof (*of Theorem 18.3*). In general there is a one-to-one correspondence between linear maps on a free modules and set maps on a basis of the free module. We define

$$F\colon \operatorname{Free}(V \times W) \to Z$$

by setting

$$F(v, w) = f(v, w)$$

and extending the definition by linearity:

$$F\Big(\sum r_i(v_i, w_i)\Big) = \sum r_i f(v_i, w_i).$$

Because f is bilinear, F vanishes on all the generators (18.1) of the subspace S. For example,

$$\begin{aligned} F\big((v_1 + v_2, w) - (v_1, w) - (v_2, w)\big) &= F(v_1 + v_2, w) - F(v_1, w) - F(v_2, w) \\ &= f(v_1 + v_2, w) - f(v_1, w) - f(v_2, w) \\ &= 0. \end{aligned}$$

So the linear map F induces a linear map of the quotient module

$$\tilde{f}\colon V\otimes W = \frac{\operatorname{Free}(V\times W)}{S} \to Z$$

such that

$$\tilde{f}(v\otimes w) = F(v,w) = f(v,w).$$

This proves the existence of the linear map $\tilde{f}$.

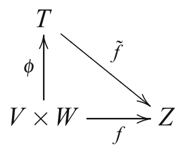
To prove uniqueness, suppose $\tilde{g}\colon V\otimes W \to Z$ is another linear map that makes the diagram (18.5) commutative. Then

$$\tilde{g}(v\otimes w) = f(v,w) = \tilde{f}(v\otimes w)$$

for all $(v,w)\in V\times W$, so $\tilde{g}$ and $\tilde{f}$ agree on decomposable elements. Since the decomposable elements span $V\otimes W$ and both $\tilde{g}$ and $\tilde{f}$ are linear, $\tilde{g}=\tilde{f}$ on all of $V\otimes W$. □

18.3 Characterization of the Tensor Product

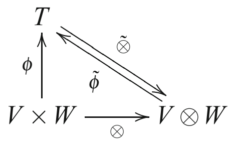
Let V and W be left modules over a commutative ring R with identity. We say that a left R-module T and an R-bilinear map $\phi\colon V\times W\to T$ have the ***universal mapping property for bilinear maps on*** $V\times W$ if given any left R-module Z and any R-bilinear map $f\colon V\times W\to Z$ there is a unique R-linear map $\tilde{f}\colon T\to Z$ making the diagram



commutative. The universal mapping property characterizes the tensor product in the following sense.

Proposition 18.4. *Suppose V, W, and T are left R-modules and $\phi\colon V\times W\to T$ an R-bilinear map satisfying the universal mapping property for bilinear maps. Then T is isomorphic to the tensor product $V\otimes W$ via the linear map $\tilde{\otimes}\colon T\to V\otimes W$.*

Proof. By the universal mapping property of T, there is a unique linear map $\tilde{\otimes}\colon T\to V\otimes W$ such that the diagram



commutes. Algebraically,

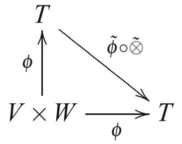
$$\tilde{\otimes}(\phi(v,w)) = v\otimes w$$

for all $(v,w)\in V\times W$.

Similarly, by the universal mapping property of $V \otimes W$, there is a map $\tilde{\phi}\colon V \otimes W \to T$ such that $\tilde{\phi}(v \otimes w) = \phi(v,w)$. Thus,

$$(\tilde{\phi} \circ \tilde{\otimes})(\phi(v,w)) = \tilde{\phi}(v \otimes w) = \phi(v,w),$$

so $\tilde{\phi} \circ \tilde{\otimes}\colon T \to T$ is a linear map that makes the diagram



commute.

On the other hand, the identity map $\mathbb{1}_T\colon T \to T$ also makes the diagram commute. By the uniqueness statement of the universal mapping property,

$$\tilde{\phi} \circ \tilde{\otimes} = \mathbb{1}_T.$$

Similarly,

$$\tilde{\otimes} \circ \tilde{\phi} = \mathbb{1}_{V \otimes W}.$$

Therefore, $\tilde{\otimes}\colon T \to V \otimes W$ is a linear isomorphism of R-modules. $\square$

As an example of the universal mapping property, we prove the following proposition.

Proposition 18.5. *Let V and W be left R-modules. There is a unique R-linear isomorphism*

$$\tilde{f}\colon V \otimes W \to W \otimes V$$

such that

$$\tilde{f}(v \otimes w) = w \otimes v$$

for all $v \in V$ and $w \in W$.

Proof. Define

$$f\colon V \times W \to W \otimes V$$

by

$$f(v,w) = w \otimes v.$$

Since f is bilinear, by the universal mapping property, there is a unique linear map $\tilde{f}\colon V \otimes W \to W \otimes V$ such that

$$(\tilde{f} \circ \otimes)(v,w) = w \otimes v,$$

or

$$\tilde{f}(v \otimes w) = w \otimes v.$$

Similarly, one can construct a unique linear map $\tilde{g}\colon W \otimes V \to V \otimes W$ such that

$$\tilde{g}(w \otimes v) = v \otimes w.$$

Then $\tilde{f}$ and $\tilde{g}$ are inverses to each other on decomposable elements $v \otimes w$. Since $V \otimes W$ is generated by decomposable elements, $\tilde{g} \circ \tilde{f} = \mathbb{1}_{V \otimes W}$ and $\tilde{f} \circ \tilde{g} = \mathbb{1}_{W \otimes V}$. □

Remark 18.6. In general, one cannot define a linear map on $V \otimes W$ by simply defining it on the decomposable elements $v \otimes w$ and then extending it by linearity. This is because an element of the tensor product $V \otimes W$ can have many different representations as a sum of decomposable elements; for example,

$$2v \otimes w = v \otimes 2w = v \otimes w + v \otimes w.$$

There is no assurance that the definition of a map on the decomposable elements will be consistent. One must always start with a bilinear map f on the Cartesian product $V \times W$. Then the universal mapping property guarantees the existence of a unique linear map $\tilde{f}$ on $V \otimes W$ with the property that $\tilde{f}(v \otimes w) = f(v, w)$.

18.4 A Basis for the Tensor Product

In this section we assume V and W to be free R-modules of finite rank with bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$, respectively.

Lemma 18.7. *For $1 \le k \le n$ and $1 \le \ell \le m$, there exist linear functions $f^{k\ell}\colon V \otimes W \to R$ such that*

$$f^{k\ell}(v_i \otimes w_j) = \delta^{(k,\ell)}_{(i,j)}$$

for all $1 \le i \le n$ and $1 \le j \le m$

Proof. Define $h^{k\ell}\colon V \times W \to R$ by setting

$$h^{k\ell}(v_i, w_j) = \delta^{(k,\ell)}_{(i,j)}$$

on the basis elements v_i of V and w_j of W and extending to $V \times W$ by bilinearity. As defined, $h^{k\ell}\colon V \times W \to R$ is bilinear, and so by the universal mapping property it corresponds to a unique linear map

$$\tilde{h}^{k\ell}\colon V \otimes W \to R$$

such that

$$\tilde{h}^{k\ell}(v_i \otimes w_j) = h^{k\ell}(v_i, w_j) = \delta^{(k,\ell)}_{(i,j)}.$$

Set $f^{k\ell} = \tilde{h}^{k\ell}$. □

Theorem 18.8. *If $v_1, \ldots, v_n$ is a basis for V and $w_1, \ldots, w_m$ is a basis for W, then*

$$\{v_i \otimes w_j \mid 1 \le i \le n, \quad 1 \le j \le m\}$$

is a basis for $V \otimes W$.

Proof. Every decomposable element $v \otimes w$ can be written as a linear combination of $v_i \otimes w_j$, for if $v = \sum a^i v_i$ and $w = \sum b^j w_j$, then

$$v \otimes w = \sum a^i b^j v_i \otimes w_j.$$

Since every element of $V \otimes W$ is a sum of decomposable elements, we see that the set $\{v_i \otimes w_j\}_{i,j}$ spans $V \otimes W$.

It remains to show that the set $\{v_i \otimes w_j\}_{i,j}$ is linearly independent. Suppose there is a linear relation

$$\sum_{i,j} c^{ij} v_i \otimes w_j = 0, \quad c^{ij} \in R.$$

Applying the linear function $f^{k\ell}$ of Lemma 18.7 gives

$$\begin{aligned} 0 = f^{k\ell}\Big(\sum_{i,j} c^{ij} v_i \otimes w_j\Big) &= \sum_{i,j} c^{ij} f^{k\ell}(v_i \otimes w_j) \\ &= \sum_{i,j} c^{ij} \delta^{(k,\ell)}_{(i,j)} \\ &= c^{k\ell}. \end{aligned}$$

This proves that the set $\{v_i \otimes w_j\}_{i,j}$ is linearly independent. □

Corollary 18.9. *If V and W are free R-modules of finite rank, then*

$$\operatorname{rk}(V \otimes W) = (\operatorname{rk} V)(\operatorname{rk} W).$$

18.5 The Dual Module

As before, R is a commutative ring with identity. For two left R-modules V and W, define $\operatorname{Hom}_R(V,W)$ to be the set of all R-linear maps $f\colon V \to W$. Under pointwise addition and scalar multiplication:

$$\begin{aligned} (f+g)(v) &= f(v) + g(v), \\ (rf)(v) &= r\big(f(v)\big), \quad f, g \in \operatorname{Hom}_R(V,W), v \in V, r \in R, \end{aligned}$$

the set $\operatorname{Hom}_R(V,W)$ becomes a left R-module. If the ring R is understood from the context, we may also write $\operatorname{Hom}(V,W)$ instead of $\operatorname{Hom}_R(V,W)$. The ***dual*** $V^\vee$ of a left R-module V is defined to be $\operatorname{Hom}_R(V,R)$.

Proposition 18.10. *Suppose V is a free R-module of rank n with basis $e_1, \dots, e_n$. Define $\alpha^i\colon V \to R$ by*

$$\alpha^i(e_j) = \delta^i_j.$$

Then the dual $V^\vee$ is a free R-module of rank n with basis $\alpha^1, \dots, \alpha^n$.

Proof. Problem 18.2. □

Corollary 18.11. *If V and W are free R-modules of rank n and m, with bases $\{v_i\}$ and $\{w_j\}$, respectively, then the functions*

$$f^{k\ell}\colon V\otimes W\to R,$$
$$f^{k\ell}(v_i\otimes w_j)=\delta^{(k,\ell)}_{(i,j)},\quad 1\le k\le n,\ 1\le \ell\le m,$$

of Lemma 18.7 constitute the dual basis for $(V\otimes W)^\vee$ dual to the basis $\{v_i\otimes w_j\}_{i,j}$ for $V\otimes W$.

18.6 Identities for the Tensor Product

This section contains a few important identities involving the tensor product. We leave some of the proofs as exercises.

Proposition 18.12. *Let V be a left R-module. Scalar multiplication*

$$f\colon R\times V\to V,$$
$$(r,v)\mapsto rv$$

is an R-bilinear map that induces an R-linear isomorphism $\tilde f\colon R\otimes_R V\to V$.

Proof. The bilinear map $f\colon R\times V\to V$ induces a linear map $\tilde f\colon R\otimes V\to V$. Define $g\colon V\to R\otimes V$ by $g(v)=1\otimes v$. Then g is R-linear and

$$g\circ\tilde f=\mathbb{1}_{R\otimes V}\quad\text{and}\quad \tilde f\circ g=\mathbb{1}_V.$$

Therefore, $\tilde f$ is an isomorphism. □

Example 18.13. For any positive integer m, it follows from the proposition that there is a $\mathbb{Z}$-isomorphism

$$\mathbb{Z}\otimes_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z})\simeq\mathbb{Z}/m\mathbb{Z}.$$

For $\alpha\in V^\vee$ and $w\in W$, denote by $\alpha(\ )w\in\operatorname{Hom}_R(V,W)$ the R-linear map that send $v\in V$ to $\alpha(v)w\in W$.

Proposition 18.14. *Let V and W be free left R-modules of finite rank. There is a unique R-linear isomorphism*

$$\tilde f\colon V^\vee\otimes W\to\operatorname{Hom}_R(V,W)$$

such that

$$\tilde f(\alpha\otimes w)=\alpha(\ )w.$$

Proof. Define $f\colon V^\vee \times W \to \mathrm{Hom}_R(V,W)$ by

$$f(\alpha, w) = \alpha(\ )w.$$

Since f is R-bilinear, it induces a unique R-linear map $\tilde{f}\colon V^\vee \otimes W \to \mathrm{Hom}(R,W)$ such that

$$\tilde{f}(\alpha \otimes w) = \alpha(\ )w.$$

Let $\{v_i\}$, $\{w_j\}$, and $\{\alpha^i\}$ be bases for V, W, and $V^\vee$ respectively. Then a basis for $V^\vee \otimes W$ is $\{\alpha^i \otimes w_j\}$ and a basis for $\mathrm{Hom}_R(V,W)$ is $\{\alpha^i(\ )w_j\}$ (Show this). Since $\tilde{f}\colon V^\vee \otimes W \to \mathrm{Hom}_R(V,W)$ takes a basis to a basis, it is an R-linear isomorphism. □

Proposition 18.15. *Let V and W be free left R-modules of finite rank. There is a unique R-linear isomorphism*

$$\tilde{f}\colon V^\vee \otimes W^\vee \to (V \otimes W)^\vee$$

such that $\tilde{f}(\alpha \otimes \beta)$ is the linear map that sends $v \otimes w$ to $\alpha(v)\beta(w)$.

Proof. Problem 18.4. □

Example 18.16 (*Tensor product of finite cyclic groups*). For any two positive integers m, n, there is a group isomorphism

$$\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/(m,n)\mathbb{Z},$$

where (m,n) is the greatest common divisor of m and n.

Proof. Define a $\mathbb{Z}$-linear map $\mathbb{Z} \to \mathbb{Z}/(m,n)\mathbb{Z}$ by $a \mapsto a \pmod{(m,n)}$. Since $m\mathbb{Z}$ is in the kernel of this map, there is an induced $\mathbb{Z}$-linear map $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/(m,n)\mathbb{Z}$. Similarly, there is a $\mathbb{Z}$-linear map $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/(m,n)\mathbb{Z}$. We can therefore define a map $f\colon \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/(m,n)\mathbb{Z}$ by

$$f(a,b) = ab \pmod{(m,n)}.$$

Clearly, f is bilinear over $\mathbb{Z}$. By the universal mapping property of the tensor product, there is a unique $\mathbb{Z}$-linear map $\tilde{f}\colon \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/(m,n)\mathbb{Z}$ such that $\tilde{f}(a \otimes b) = ab \pmod{(m,n)}$.

By the third defining property of the tensor product (18.4), $a \otimes b = ab(1 \otimes 1)$. Thus, the $\mathbb{Z}$-module $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$ is cyclic with generator $1 \otimes 1$. It remains to determine its order. It is well known from number theory that there exist integers x and y such that

$$(m,n) = mx + ny.$$

Since

$$\begin{aligned} (m,n)(1 \otimes 1) &= mx(1 \otimes 1) + ny(1 \otimes 1) \\ &= mx \otimes 1 + 1 \otimes ny = 0, \end{aligned}$$

the order of $1 \otimes 1$ is a factor of (m,n).

Suppose d is a positive factor of (m,n), but $d < (m,n)$. Then $f\big(d(1\otimes 1)\big) = f(d\otimes 1) = d \neq 0$ in $\mathbb{Z}/(m,n)\mathbb{Z}$. Hence, $d(1\otimes 1) \neq 0$. This proves that $1\otimes 1$ has order (m,n) in $\mathbb{Z}/m\mathbb{Z}\otimes\mathbb{Z}/n\mathbb{Z}$. Thus, $\mathbb{Z}/m\mathbb{Z}\otimes\mathbb{Z}/n\mathbb{Z}$ is cyclic of order (m,n). □

In particular, if m and n are relatively prime, then

$$\mathbb{Z}/m\mathbb{Z}\otimes\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/\mathbb{Z} = 0.$$

18.7 Functoriality of the Tensor Product

In this section we show that the tensor product gives rise to a functor.

Proposition 18.17. *Let $f\colon V \to V'$ and $g\colon W \to W'$ be linear maps of left R-modules. Then there is a unique R-linear map*

$$f\otimes g\colon V\otimes W \to V'\otimes W'$$

such that $(f\otimes g)(v\otimes w) = f(v)\otimes g(w)$.

Proof. Define

$$h\colon V\times W \to V'\otimes W'$$

by

$$h(v,w) = f(v)\otimes g(w).$$

This h is clearly an R-bilinear map. By the universal mapping property (Theorem 18.3), there is a unique R-linear map

$$\tilde{h}\colon V\otimes W \to V'\otimes W'$$

such that

$$\tilde{h}(v\otimes w) = h(v,w) = f(v)\otimes g(w).$$

The map $\tilde{h}$ is our $f\otimes g$. □

Thus, the tensor product construction associates to a pair of left R-modules (V,W) their tensor product $V\otimes W$, and to a pair of R-linear maps $(f\colon V\to V', g\colon W\to W')$ the R-linear map

$$f\otimes g\colon V\otimes W \to V'\otimes W'.$$

It is not difficult to check that this construction satisfies the two properties of a functor:

(i) If $\mathbb{1}_V\colon V\to V$ and $\mathbb{1}_W\colon W\to W$ are identity maps, then

$$\mathbb{1}_V\otimes\mathbb{1}_W\colon V\otimes W \to V\otimes W$$

is the identity map on $V\otimes W$.

(ii) If $(f\colon V\to V', g\colon W\to W')$ and $(f'\colon V'\to V'', g'\colon W'\to W'')$ are pairs of R-linear maps, then

$$(f'\otimes g')\circ(f\otimes g) = (f'\circ f)\otimes(g'\circ g).$$

So the tensor product is a functor from the category of pairs of R-modules and pairs of linear maps to the category of R-modules and linear maps.

18.8 Generalization to Multilinear Maps

The tensor product construction can be generalized to an arbitrary number of factors. For example, if U, V, and W are left R-modules, we can construct the tensor product $U \otimes V \otimes W$ by following the same procedure as before. First, form the free R-module $\mathrm{Free}(U \times V \times W)$ with basis elements all $(u,v,w) \in U \times V \times W$. Then let S be the R-submodule of $\mathrm{Free}(U \times V \times W)$ spanned by all "trilinear relations":

$$\begin{aligned} &(u_1+u_2,v,w)-(u_1,v,w)-(u_2,v,w),\\ &(ru,v,w)-r(u,v,w), \quad \text{and so on.} \end{aligned}$$

The tensor product $U \otimes V \otimes W$ is defined to be the quotient module $\mathrm{Free}(U \times V \times W)/S$. The natural map

$$\begin{aligned} \otimes \colon U \times V \times W &\to U \otimes V \otimes W \\ (u,v,w) &\mapsto u \otimes v \otimes w \end{aligned}$$

is trilinear and satisfies the universal mapping property for trilinear maps on $U \times V \times W$.

18.9 Associativity of the Tensor Product

Proposition 18.18. *If U, V, and W are left R-modules, then there is a unique R-linear isomorphism*

$$U \otimes (V \otimes W) \to U \otimes V \otimes W$$

that sends $u \otimes (v \otimes w)$ to $u \otimes v \otimes w$ for all $u \in U$, $v \in V$ and $w \in W$.

Proof. It suffices to prove that $U \otimes (V \otimes W)$ together with the trilinear map

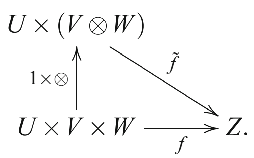
$$\begin{aligned} \phi \colon U \times V \times W &\to U \otimes (V \otimes W) \\ (u,v,w) &\mapsto u \otimes (v \otimes w) \end{aligned}$$

satisfies the universal mapping property for trilinear maps on $U \times V \times W$. As in Proposition 18.4, any two spaces satisfying the universal mapping property for trilinear maps on $U \times V \times W$ are isomorphic.

Let Z be any vector space and $f \colon U \times V \times W \to Z$ a trilinear map. Fix $u \in U$. Since $f(u,v,w)$ is bilinear in v and w, there is a unique linear map $\tilde{f}(u, \ ) \colon V \otimes W \to Z$ such that

$$\tilde{f}(u, v \otimes w) = f(u,v,w).$$

Thus, there is a commutative diagram



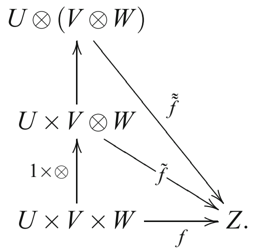
Now $\tilde{f}$ is bilinear in its two arguments, so there is a unique linear map

$$\tilde{\tilde{f}}\colon U \otimes (V \otimes W) \to U \otimes V \otimes W$$

such that

$$\tilde{\tilde{f}}(u \otimes (v \otimes w)) = \tilde{f}(u, v \otimes w) = f(u, v, w).$$

Hence, $\phi\colon U \times V \times W \to U \otimes (V \otimes W)$ satisfies the universal mapping property for trilinear maps on $U \times V \times W$, and there is a commutative diagram



Taking $Z = U \otimes V \otimes W$ and $f = \otimes\colon U \times V \times W \to U \otimes V \otimes W$, we get

$$\tilde{\tilde{f}}\colon U \otimes (V \otimes W) \to U \otimes V \otimes W$$

such that

$$\tilde{\tilde{f}}(u \otimes (v \otimes w)) = f(u, v, w) = u \otimes v \otimes w.$$

By the trilinear analogue of Proposition 18.4, $\tilde{\tilde{f}}$ is an isomorphism. □

In the same way we see that there is an isomorphism

$$(U \otimes V) \otimes W \to U \otimes V \otimes W$$

that takes $(u \otimes v) \otimes w$ to $u \otimes v \otimes w$.

Corollary 18.19. *If U, V, and W are left R-modules, then there is a unique R-linear isomorphism*

$$U \otimes (V \otimes W) \xrightarrow{\sim} (U \otimes V) \otimes W$$

such that

$$u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w$$

for all $u \in U$, $v \in V$ and $w \in W$.

18.10 The Tensor Algebra

For a left R-module V, we define

$$T^0(V) = R, \quad T^1(V) = V, \quad T^2(V) = V \otimes V, \; \dots \,,$$
$$T^k(V) = \underbrace{V \otimes \cdots \otimes V}_{k}, \; \dots \,,$$

and

$$T(V) = \bigoplus_{k=0}^{\infty} T^k(V).$$

Here the direct sum $\bigoplus$ means that each element v in $T(V)$ is uniquely a ***finite*** sum $v = \sum_i v_{k_i}$, where $v_{k_i} \in T^{k_i}(V)$. Elements of $T^k(V)$ are said to be ***homogeneous*** of degree k. There is a multiplication map on $T(V)$: first define

$$\mu : T^k(V) \times T^\ell(V) \to T^{k+\ell}(V),$$
$$(x, y) \mapsto x \otimes y,$$

and then extend μ to $T(V) \times T(V) \to T(V)$ by R-bilinearity. In this way, $T(V)$ becomes a graded R-algebra, called the ***tensor algebra*** of V. By Corollary 18.19, the tensor algebra $T(V)$ is associative.

Problems

In the following problems, R is a commutative ring with identity.

18.1. Free R-modules of rank n
Show that a free R-module of rank n is isomorphic to $R \oplus \cdots \oplus R$ (n copies).

18.2. The dual of a free module
Prove Proposition 18.10.

18.3. Hom of free modules
Prove that if V and W are free left R-modules of rank n and m, respectively, then $\operatorname{Hom}_R(V, W)$ is a free R-module of rank mn.

18.4. Dual of a tensor product
Prove Proposition 18.15.

18.5. Tensor product of finite cyclic groups
Let m and n be two positive integers. Identify $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ as a finite abelian group.

§19 The Exterior Power

In this section, R denotes a commutative ring with identity and all R-modules are left R-modules. Let V be a left R-module and let $V^k = V \times \cdots \times V$ be the Cartesian product of k copies of V. An ***alternating k-linear map*** from V to R is a k-linear map $f\colon V^k \to R$ that vanishes whenever two of the arguments are equal, i.e.,

$$f(\cdots, v, \cdots, v, \cdots) = 0.$$

Just as the tensor product solves the universal mapping problem for multilinear maps over R, so we will now construct a module that solves the universal mapping problem for alternating multilinear maps over R. The question is, given an R-module V, does there exist an R-module W such that alternating k-linear maps from V^k to an R-module Z correspond canonically in a one-to-one way to *linear* maps from W to Z? The construction of the exterior power $\bigwedge^k V$ answers the question in the affirmative.

19.1 The Exterior Algebra

For a left R-module V, let $I(V)$ be the two-sided ideal generated by all elements of the form $v \otimes v$ in the tensor algebra $T(V)$, for $v \in V$. So $I(V)$ includes elements such as $a \otimes v \otimes v \otimes b$ for $a, b \in T(V)$.

Definition 19.1. The ***exterior algebra*** $\bigwedge(V)$ is defined to be the quotient algebra $T(V)/I(V)$. We denote the image of $v_1 \otimes \cdots \otimes v_k$ by

$$v_1 \wedge \cdots \wedge v_k.$$

Such an element is said to be ***decomposable*** and the operation $\wedge$ is called the ***wedge product***.

Define $\bigwedge^k(V)$, the kth ***exterior power*** of V, to be the image of $T^k(V)$ in $\bigwedge(V)$ under the projection $T(V) \to \bigwedge(V)$. Then there is a canonical module isomorphism

$$\bigwedge^k(V) \simeq \frac{T^k(V)}{T^k(V) \cap I(V)} = \frac{T^k(V)}{I^k(V)},$$

where $I^k(V) := T^k(V) \cap I(V)$ consists of the homogeneous elements of degree k in $I(V)$. In this way the exterior algebra $\bigwedge(V)$ inherits a grading from the tensor algebra and itself becomes a graded R-algebra. We often write $\bigwedge^k V$ for $\bigwedge^k(V)$.

19.2 Properties of the Wedge Product

Let R be a commutative ring with identity and V a left R-module. In this subsection, we derive properties of the wedge product on the exterior algebra $\bigwedge(V)$.

Lemma 19.2. *Let V be an R-module. In the exterior algebra $\bigwedge(V)$,*

$$u \wedge v = -v \wedge u$$

for all $u, v \in V$.

Proof. For any $w \in V$, we have by definition $w \otimes w \in I(V)$, so that in the exterior algebra $\bigwedge(V)$, the wedge product $w \wedge w$ is 0. Thus, for all $u, v \in V$,

$$\begin{aligned} 0 &= (u+v) \wedge (u+v) \\ &= u \wedge u + u \wedge v + v \wedge u + v \wedge v \\ &= u \wedge v + v \wedge u. \end{aligned}$$

Hence, $u \wedge v = -v \wedge u$. □

Proposition 19.3. *If $u \in \bigwedge^k V$ and $v \in \bigwedge^\ell V$, then $u \wedge v \in \bigwedge^{k+\ell} V$ and*

$$u \wedge v = (-1)^{k\ell} v \wedge u. \tag{19.1}$$

Proof. Since both sides of (19.1) are linear in u and in v, it suffices to prove the equation for decomposable elements.

So suppose

$$u = u_1 \wedge \cdots \wedge u_k \quad \text{and} \quad v = v_1 \wedge \cdots \wedge v_\ell.$$

By Lemma 19.2, $v_1 \wedge u_i = -u_i \wedge v_1$. In $u \wedge v$, to move v_1 across $u_1 \wedge \cdots \wedge u_k$ requires k adjacent transpositions and introduces a sign of $(-1)^k$. Similarly, moving v_2 across $u_1 \wedge \cdots \wedge u_k$ also introduces a sign of $(-1)^k$. Hence,

$$\begin{aligned} u \wedge v &= (-1)^k v_1 \wedge u_1 \wedge \cdots \wedge u_k \wedge v_2 \wedge \cdots \wedge v_\ell \\ &= (-1)^k (-1)^k v_1 \wedge v_2 \wedge u_1 \wedge \cdots \wedge u_k \wedge v_3 \wedge \cdots \wedge v_\ell \\ &\ \ \vdots \\ &= (-1)^{k\ell} v_1 \wedge \cdots \wedge v_\ell \wedge u_1 \wedge \cdots \wedge u_k \\ &= (-1)^{k\ell} v \wedge u. \end{aligned}$$ □

Lemma 19.4. *In a decomposable element $v_1 \wedge \cdots \wedge v_k \in \bigwedge^k V$, with each $v_i \in V$, a transposition of v_i and v_j introduces a minus sign:*

$$v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k = -v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_k.$$

Proof. Without loss of generality, we may assume $i < j$. Let $a = v_{i+1} \wedge \cdots \wedge v_{j-1}$. It suffices to prove that

$$v_i \wedge a \wedge v_j = -v_j \wedge a \wedge v_i.$$

By Proposition 19.3,

$$\begin{aligned} (v_i \wedge a) \wedge v_j &= (-1)^{\deg a + 1} v_j \wedge (v_i \wedge a) \\ &= (-1)^{\deg a + 1} (-1)^{\deg a} v_j \wedge (a \wedge v_i). \end{aligned}$$ □

Proposition 19.5. *If π is a permutation on k letters and $v_i \in V$, then*

$$v_{\pi(1)} \wedge \cdots \wedge v_{\pi(k)} = (\operatorname{sgn} \pi) v_1 \wedge \cdots \wedge v_k.$$

Proof. Suppose π can be written as a product of ℓ transpositions. Then $\operatorname{sgn}(\pi) = (-1)^\ell$. Since each transposition of the subscripts introduces a minus sign,

$$v_{\pi(1)} \wedge \cdots \wedge v_{\pi(k)} = (-1)^\ell v_1 \wedge \cdots \wedge v_k = (\operatorname{sgn} \pi)\, v_1 \wedge \cdots \wedge v_k. \qquad \square$$

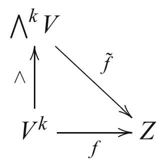
19.3 Universal Mapping Property for Alternating k-Linear Maps

If V is a left R-module, V^k is the Cartesian product $V \times \cdots \times V$ of k copies of V, and $\bigwedge^k V$ is the kth exterior power of V, then there is a natural map $\wedge\colon V^k \to \bigwedge^k V$,

$$\wedge(v_1, \ldots, v_k) = v_1 \wedge \cdots \wedge v_k.$$

This map is clearly k-linear. By Proposition 19.5, it is also alternating.

Theorem 19.6 (Universal mapping property for alternating k-linear maps). *For any R-module Z and any alternating k-linear map $f\colon V^k \to Z$ over R, there is a unique linear map $\tilde{f}\colon \bigwedge^k V \to Z$ over R such that the diagram*



commutes.

The commutativity of the diagram is equivalent to

$$\tilde{f}(v_1 \wedge \cdots \wedge v_k) = f(v_1, \ldots, v_k) \tag{19.2}$$

for all $v_1, \ldots, v_k$ in V.

Proof. Since f is k-linear, by the universal mapping property of the tensor product, there is a unique linear map $h\colon T^k V \to Z$ such that

$$h(v_1 \otimes \cdots \otimes v_k) = f(v_1, \ldots, v_k).$$

Since f is alternating,

$$h(\cdots \otimes v \otimes v \otimes \cdots) = f(\ldots, v, v, \ldots) = 0 \quad \text{for all } v \in V.$$

So h vanishes on the submodule $I^k(V)$, and therefore h induces a linear map on the quotient module

$$\tilde{f}\colon \frac{T^k V}{I^k V} = \bigwedge^k V \to Z$$

such that

$$\tilde{f}(v_1 \wedge \cdots \wedge v_k) = h(v_1 \otimes \cdots \otimes v_k) = f(v_1, \ldots, v_k).$$

This proves the existence of $\tilde{f}\colon \bigwedge^k V \to Z$.

To prove its uniqueness, note that (19.2) defines $\tilde{f}$ on all decomposable elements in $\bigwedge^k V$. Since every element of $\bigwedge^k V$ is a sum of decomposable elements and $\tilde{f}$ is linear, (19.2) determines uniquely the value of $\tilde{f}$ on all of $\bigwedge^k V$. □

For a module V over a commutative ring R with identity, denote by $L_k(V)$ the R-module of k-linear maps from V^k to R, and $A_k(V)$ the R-module of alternating k-linear maps from V^k to R. By the universal mapping property for the tensor product, k-linear maps on V^k may be identified with linear maps on the tensor power T^kV:

$$L_k(V) \simeq \left(T^kV\right)^\vee,$$
$$f \mapsto \tilde{f},$$

where

$$f(v_1, \ldots, v_k) = \tilde{f}(v_1 \otimes \cdots \otimes v_k).$$

Similarly, by the universal mapping property for the exterior power, alternating k-linear maps on V may be identified with linear maps on $\bigwedge^k V$:

$$A_k(V) \simeq \left(\bigwedge^k V\right)^\vee, \tag{19.3}$$
$$f \mapsto \tilde{f},$$

where

$$f(v_1, \ldots, v_k) = \tilde{f}(v_1 \wedge \cdots \wedge v_k).$$

19.4 A Basis for $\bigwedge^k V$

Let R be a commutative ring with identity. We prove in this section that if V is a free R-module of finite rank, then for any k the exterior power $\bigwedge^k V$ is also a free R-module of finite rank. Moreover, from a basis for V, we construct a basis for $\bigwedge^k V$.

Lemma 19.7. *If $B = \{e_1, \ldots, e_n\}$ is a basis for the free R-module V, then*

$$e_1 \wedge \cdots \wedge e_n \neq 0.$$

It follows that for $1 \le i_1 < \cdots < i_k \le n$, we have $e_{i_1} \wedge \cdots \wedge e_{i_k} \neq 0$.

Proof. It will be enough to define a linear map $\bigwedge^n V \to V$ that takes $e_1 \wedge \cdots \wedge e_n$ to a nonzero element. First define an alternating set map $f\colon B^n \to R$ by

$$f(e_{\pi(1)}, \ldots, e_{\pi(n)}) = \operatorname{sgn}(\pi),$$

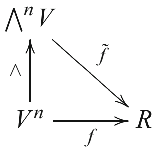
where π is a permutation of $\{1, \ldots, n\}$. Next extend f to an n-linear map $f\colon V^n \to R$, which is also alternating because it is alternating on basis elements. Indeed,

$$
\begin{aligned}
f\Big(\ldots,\sum a^i e_i,\ldots,\sum b^j e_j,\ldots\Big) \\
&= \sum a^i b^j f(\ldots,e_i,\ldots,e_j,\ldots) && \text{(by } n\text{-linearity)} \\
&= -\sum a^i b^j f(\ldots,e_j,\ldots,e_i,\ldots) && \text{(by definition of } f\text{)} \\
&= -f\Big(\ldots,\sum b^j e_j,\ldots,\sum a^i e_i,\ldots\Big) && \text{(by } n\text{-linearity)}
\end{aligned}
$$

By the universal mapping property for alternating n-linear maps, there is a unique linear map

$$\tilde{f}\colon \bigwedge^n V \to R$$

such that the diagram



commutes. In particular,

$$\tilde{f}(e_1 \wedge \cdots \wedge e_n) = f(e_1,\ldots,e_n) = 1.$$

This shows that $e_1 \wedge \cdots \wedge e_n \neq 0$. □

Theorem 19.8. *If* $\{e_1,\ldots,e_n\}$ *is a basis for* V*, then*

$$S := \{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\}$$

is a basis for $\bigwedge^k V$.

Proof. Since the exterior power $\bigwedge^k V$ is a quotient of the tensor power $T^k V$ and

$$\{e_{i_1} \otimes \cdots \otimes e_{i_k} \mid 1 \le i_1,\ldots,i_k \le n\}$$

is a basis for $T^k V$, the set S spans $\bigwedge^k V$.

It remains to show that the set S is linearly independent. We introduce the multi-index notation $I = (1 \le i_1 < \cdots < i_k \le n)$ and

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

Suppose there is a linear relation

$$\sum a^I e_I = 0, \tag{19.4}$$

where I runs over all multi-indices $1 \le i_1 < \cdots < i_k \le n$ of length k. If $J = (j_1 < \cdots < j_k)$ is one particular multi-index in this sum, let J' be its complement, also arranged in increasing order. By definition, e_J and $e_{J'}$ have no factor in common, but if I and J both have length k and $I \neq J$, then there is an e_i in e_I that is not in e_J, so that e_I and $e_{J'}$ will have e_i in common. Hence,

$$\begin{cases} e_J \wedge e_{J'} = \pm e_1 \wedge \cdots \wedge e_n, & \\ e_I \wedge e_{J'} = 0, & \text{if } I \neq J. \end{cases}$$

Taking the wedge product of (19.4) with $e_{J'}$ gives

$$0 = \left(\sum a^I e_I\right) \wedge e_{J'} = a^J e_J \wedge e_{J'} = \pm a^J e_1 \wedge \cdots \wedge e_n.$$

Since $e_1 \wedge \cdots \wedge e_n \neq 0$ (Lemma 19.7), $a^J = 0$. So the set S is linearly independent and hence is a basis for the exterior power $\bigwedge^k V$. □

Corollary 19.9. *If V is a free R-module of rank n, then $\bigwedge^k V$ is a free R-module of rank $\binom{n}{k}$.*

19.5 Nondegenerate Pairings

Let R be a commutative ring with identity. A ***pairing*** of two R-modules V and W is a bilinear map

$$\langle\ ,\ \rangle \colon V \times W \to R.$$

The pairing is said to be ***nondegenerate*** if

$$\langle v, w\rangle = 0 \text{ for all } w \in W \quad \Rightarrow \quad v = 0,$$

and

$$\langle v, w\rangle = 0 \text{ for all } v \in V \quad \Rightarrow \quad w = 0.$$

Example 19.10. An inner product on a real vector space V is a nondegenerate pairing of V with itself, since if $\langle v, w\rangle = 0$ for all $w \in V$, then $\langle v, v\rangle = 0$ and hence $v = 0$. Similarly, $\langle v, w\rangle = 0$ for all $v \in V$ implies that $\langle w, w\rangle = 0$ and hence $w = 0$.

If there is a pairing $\langle\ ,\ \rangle$ between two left R-modules V and W, then each v in V defines a linear map

$$\langle v,\ \rangle \colon W \to R.$$

So the pairing induces a map $V \to W^\vee$, given by

$$v \mapsto \langle v,\ \rangle.$$

This map is clearly R-linear. Similarly, the pairing also induces an R-linear map $W \to V^\vee$ via $w \mapsto \langle\ , w\rangle$. The definition of nondegeneracy says precisely that the two induced linear maps $V \to W^\vee$ and $W \to V^\vee$ are injective. For finite-dimensional vector spaces this is enough to imply isomorphism.

Lemma 19.11. *Let V and W be finite-dimensional vector spaces over a field R. If $\langle\ ,\ \rangle \colon V \times W \to R$ is a nondegenerate pairing, then the induced maps $V \to W^\vee$ and $W \to V^\vee$ are isomorphisms.*

Proof. By the injectivity of the induced linear maps $V \to W^\vee$ and $W \to V^\vee$,

$$\dim V \le \dim W^\vee \quad \text{and} \quad \dim W \le \dim V^\vee.$$

Since a finite-dimensional vector space and its dual have the same dimension,

$$\dim V \le \dim W^\vee = \dim W \le \dim V^\vee = \dim V.$$

Hence, $\dim V = \dim W$. So the injections $V \to W^\vee$ and $W \to V^\vee$ are both isomorphisms. □

Any finite-dimensional vector space V is isomorphic to its dual $V^\vee$, because they have the same dimension, but in general there is no canonical isomorphism between V and $V^\vee$. If V is finite-dimensional and has a nondegenerate pairing, however, then the pairing induces a canonical isomorphism

$$\begin{aligned} V &\to V^\vee, \\ v &\mapsto \langle v, \ \rangle. \end{aligned}$$

19.6 A Nondegenerate Pairing of $\bigwedge^k(V^\vee)$ with $\bigwedge^k V$

Let V be a vector space. In this subsection we establish a canonical isomorphism between $\bigwedge^k(V^\vee)$ and $\left(\bigwedge^k V\right)^\vee$ by finding a nondegenerate pairing of of $\bigwedge^k(V^\vee)$ with $\bigwedge^k V$.

Proposition 19.12. *Let V be a module over a commutative ring R with identity. The multilinear map*

$$\begin{aligned} (V^\vee)^k \times V^k &\to R, \\ \Big((\beta^1,\dots,\beta^k),(v_1,\dots,v_k)\Big) &\mapsto \det[\beta^i(v_j)] \end{aligned}$$

induces a pairing

$$\bigwedge^k(V^\vee) \times \bigwedge^k V \to R.$$

Proof. For a fixed $(\beta^1,\dots,\beta^k) \in (V^\vee)^k$, the function: $V^k \to R$

$$(v_1,\dots,v_k) \mapsto \det[\beta^i(v_j)]$$

is alternating and k-linear. By the universal mapping property for $\bigwedge^k V$, there is a unique linear map $\bigwedge^k V \to R$ such that

$$v_1 \wedge \cdots \wedge v_k \mapsto \det[\beta^i(v_j)].$$

In this way we have constructed a map

$$h\colon (V^\vee)^k \to \left(\bigwedge^k V\right)^\vee$$

such that

$$h(\beta^1,\ldots,\beta^k)(v_1\wedge\cdots\wedge v_k)=\det[\beta^i(v_j)].$$

Since $h(\beta^1,\ldots,\beta^k)$ is alternating k-linear in $\beta^1,\ldots,\beta^k$, by the universal mapping property again, there is a linear map

$$\tilde{h}\colon \bigwedge^k(V^\vee)\to\left(\bigwedge^k V\right)^\vee$$

such that

$$\tilde{h}(\beta^1\wedge\cdots\wedge\beta^k)(v_1\wedge\cdots\wedge v_k)=\det[\beta^i(v_j)].$$

The map $\tilde{h}$ gives rise to a bilinear map

$$\bigwedge^k(V^\vee)\times\bigwedge^k V\to R$$

such that

$$(\beta^1\wedge\cdots\wedge\beta^k, v_1\wedge\cdots\wedge v_k)\mapsto\det[\beta^i(v_j)].$$

□

Theorem 19.13. *If V is a free R-module of finite rank n, then for any positive integer $k\le n$, the linear map*

$$\tilde{h}\colon \bigwedge^k(V^\vee)\to\left(\bigwedge^k V\right)^\vee,$$
$$(\beta^1\wedge\cdots\wedge\beta^k)\mapsto(v_1\wedge\cdots\wedge v_k\mapsto\det[\beta^i(v_j)]),\tag{19.5}$$

from the proof of Proposition 19.12 is an isomorphism.

Proof. To simplify the notation, we will rename as f the map $\tilde{h}$. Since duality leaves the rank of a free module unchanged, by Corollary 19.9 both $\bigwedge^k(V^\vee)$ and $(\bigwedge^k V)^\vee$ are free R-modules of rank $\binom{n}{k}$. We will prove that f maps a basis for $\bigwedge^k(V^\vee)$ to a basis for $(\bigwedge^k V)^\vee$, and is therefore an isomorphism.

To this end, let $e_1,\ldots,e_n$ be a basis for V, and $\alpha^1,\ldots,\alpha^n$ its dual basis for $V^\vee$. Then

$$\{e_I:=e_{i_1}\wedge\cdots\wedge e_{i_k}\mid 1\le i_1<\cdots<i_k\le n\}$$

is a basis for $\bigwedge^k V$, and

$$\{\alpha^I:=\alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k}\mid 1\le i_1<\cdots<i_k\le n\}$$

is a basis for $\bigwedge^k(V^\vee)$. Denote by e_I^* the basis for $\left(\bigwedge^k V\right)^\vee$ dual to $\{e_I\}$, defined by

$$e_I^*(e_J)=\delta_{I,J}.$$

We claim that $f(\alpha^I)=e_I^*$. First consider $f(\alpha^I)(e_J)$ for $I=J$:

$$\begin{aligned}f(\alpha^I)(e_I)&=f(\alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k})(e_{i_1}\wedge\cdots\wedge e_{i_k})\\&=\det[\alpha^{i_r}(e_{i_s})]=\det[\delta^r_s]=1.\end{aligned}$$

Next, if $I \neq J$, then some i_{r_o} is different from $j_1, \dots, j_k$. Then

$$f(\alpha^I)(e_J) = \det[\alpha^{i_r}(e_{j_s})] = 0,$$

because the r_o-th row is identically zero. Thus, $f(\alpha^I) = e_I^*$ for all multi-indices I. Since f maps a basis to a basis, it is an isomorphism of free R-modules. □

Theorem 19.13 and (19.3) together give a sequence of module isomorphisms

$$\bigwedge^k (V^\vee) \simeq \left(\bigwedge^k V\right)^\vee \simeq A_k(V). \tag{19.6}$$

19.7 A Formula for the Wedge Product

In this section, V is a free module of finite rank over a commutative ring R with identity. According to (19.6), an element $\beta^1 \wedge \cdots \wedge \beta^k$ in $\bigwedge^k(V^\vee)$ can be interpreted as a k-linear map $V \times \cdots \times V \to R$. Thus, if $\alpha \in \bigwedge^k(V^\vee)$ and $\beta \in \bigwedge^\ell(V^\vee)$, then

$$\alpha \wedge \beta \in \bigwedge^{k+\ell}(V^\vee) \simeq A_{k+\ell}(V)$$

can be interpreted as a $(k+\ell)$-linear map on V. We will now identify this $(k+\ell)$-linear map $\alpha \wedge \beta \colon V^{k+\ell} \to R$.

Definition 19.14. A (k,ℓ)-***shuffle*** is a permutation $\pi \in S_{k+\ell}$ such that

$$\pi(1) < \cdots < \pi(k) \quad \text{and} \quad \pi(k+1) < \cdots < \pi(k+\ell).$$

Proposition 19.15. *For $\alpha \in \bigwedge^k(V^\vee)$, $\beta \in \bigwedge^\ell(V^\vee)$, and $v_i \in V$,*

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+\ell}) = \sum_\pi \operatorname{sgn}(\pi) \alpha\left(v_{\pi(1)}, \dots, v_{\pi(k)}\right) \beta\left(v_{\pi(k+1)}, \dots, v_{\pi(k+\ell)}\right),$$

where π runs over all the (k,ℓ)-shuffles.

Proof. Since both sides are linear in α and in β, we may assume that both α and β are decomposable, say

$$\alpha = \alpha^1 \wedge \cdots \wedge \alpha^k, \quad \beta = \beta^1 \wedge \cdots \wedge \beta^\ell.$$

Given any permutation

$$\rho = \begin{bmatrix} 1 & \cdots & k & k+1 & \cdots & k+\ell \\ \rho(1) & \cdots & \rho(k) & \rho(k+1) & \cdots & \rho(k+\ell) \end{bmatrix} \in S_{k+\ell},$$

we can turn it into a (k,ℓ)-shuffle as follows. Arrange $\rho(1), \dots, \rho(k)$ in increasing order

$$\rho(i_1) < \cdots < \rho(i_k)$$

and define $\sigma(j) = i_j$ for $1 \le j \le k$. Similarly, arrange $\rho(k+1),\dots,\rho(k+\ell)$ in increasing order

$$\rho(i_{k+1}) < \cdots < \rho(i_{k+\ell})$$

and define $\tau(j) = i_j$ for $k+1 \le j \le k+\ell$. Then $\rho \circ \sigma \circ \tau$ is a (k,ℓ)-shuffle.

Thus, we can get all permutations of $1,\dots,k+\ell$ by taking all (k,ℓ)-shuffles and permuting separately the first k elements and the last ℓ elements. Let S_ℓ denote the group of permutations of $k+1,\dots,k+\ell$. Then using (19.5),

$$\begin{aligned}
&(\alpha^1 \wedge \cdots \wedge \alpha^k \wedge \beta^1 \wedge \cdots \wedge \beta^\ell)(v_1,\dots,v_{k+\ell}) \\
&\quad = \sum_{\rho \in S_{k+\ell}} \operatorname{sgn}(\rho)\alpha^1\left(v_{\rho(1)}\right)\cdots\alpha^k\left(v_{\rho(k)}\right)\beta^1\left(v_{\rho(k+1)}\right)\cdots\beta^\ell\left(v_{\rho(k+\ell)}\right) \\
&\quad = \sum_{(k,\ell)\text{-shuffles } \pi} \sum_{\sigma \in S_k} \sum_{\tau \in S_\ell} \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \\
&\qquad \alpha^1\left(v_{\pi(\sigma(1))}\right)\cdots\alpha^k\left(v_{\pi(\sigma(k))}\right)\beta^1\left(v_{\pi(\tau(k+1))}\right)\cdots\beta^k\left(v_{\pi(\tau(k+\ell))}\right). \qquad (19.7)
\end{aligned}$$

(The sign comes from the fact that $\rho = \pi \circ \tau \circ \sigma$.)

Now set $w_i = v_{\pi(i)}$. Then $w_{\sigma(j)} = v_{\pi(\sigma(j))}$, so

$$\begin{aligned}
&\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\alpha^1\left(v_{\pi(\sigma(1))}\right)\cdots\alpha^k\left(v_{\pi(\sigma(k))}\right) \\
&\quad = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\alpha^1\left(w_{\sigma(1)}\right)\cdots\alpha^k\left(w_{\sigma(k)}\right) \\
&\quad = (\alpha^1 \wedge \cdots \wedge \alpha^k)(w_1,\dots,w_k) \\
&\quad = (\alpha^1 \wedge \cdots \wedge \alpha^k)\left(v_{\pi(1)},\dots,v_{\pi(k)}\right).
\end{aligned}$$

Thus, the sum (19.7) is

$$\begin{aligned}
&\sum_{(k,\ell)\text{-shuffles } \pi} \operatorname{sgn}(\pi)(\alpha^1 \wedge \cdots \wedge \alpha^k)\left(v_{\pi(1)},\dots,v_{\pi(k)}\right) \\
&\qquad\qquad \cdot (\beta^1 \wedge \cdots \wedge \beta^\ell)\left(v_{\pi(k+1)},\dots,v_{\pi(k+\ell)}\right) \\
&\quad = \sum_{(k,\ell)\text{-shuffles } \pi} \operatorname{sgn}(\pi)\alpha\left(v_{\pi(1)},\dots,v_{\pi(k)}\right)\beta\left(v_{\pi(k+1)},\dots,v_{\pi(k+\ell)}\right). \qquad \square
\end{aligned}$$

Problems

In the following problems, let V be a left module over a commutative ring R with identity.

19.1. Symmetric power of an R-module
A k-linear map $f\colon V^k \to R$ is ***symmetric*** if

$$f(v_{\pi(1)},\dots,v_{\pi(k)}) = f(v_1,\dots,v_k)$$

for all permutations $\pi \in S_k$ and all $v_i \in V$.

(a) Mimicking the definition of the exterior power $\bigwedge^k(V)$, define the symmetric power $S^k(V)$.
(b) State and prove a universal mapping property for symmetric k-linear maps over R.

§20 Operations on Vector Bundles

On a compact oriented Riemannian manifold M of dimension 2, let Ω be the curvature matrix of the Riemannian connection relative to an orthonormal frame e_1, e_2 on an open subset of M. In the proof of the Gauss–Bonnet theorem, we saw that the 2-form $K\,\mathrm{vol}$ on M is locally the entry Ω^1_2 of the curvature matrix. The integral $\int_M K\,\mathrm{vol}$ turns out to be a topological invariant. There are many ways to generalize the Gauss–Bonnet theorem; for example, we can (i) replace the 2-manifold M by a higher-dimensional manifold and replace $K\,\mathrm{vol}$ by some other differential form constructed from the curvature form of M, or (ii) replace the tangent bundle by an arbitrary vector bundle. To carry out these generalizations we will need some preliminaries on vector bundles.

In this section, we define subbundles, quotient bundles, and pullback bundles. Starting from two C^∞ vector bundles E and F over the same base manifold M, we construct their direct sum $E \oplus F$ over M; it is a vector bundle whose fiber at a point $x \in M$ is the direct sum of the fibers E_x and F_x. This construction can be generalized to any smooth functor of vector spaces, so that one can obtain similarly C^∞ vector bundles $E \otimes F$, $\mathrm{Hom}(E,F)$, $E^\vee$, and $\bigwedge^k E$ over M (see Section 20.7).

20.1 Vector Subbundles

Definition 20.1. A C^∞ ***subbundle*** of a C^∞ vector bundle $\pi\colon E \to M$ is a C^∞ vector bundle $\rho\colon F \to M$ such that

(i) F is a regular submanifold of E, and
(ii) the inclusion map $i\colon F \to E$ is a bundle homomorphism.

By composing a section of the subbundle F with the inclusion map $i\colon F \to E$, we may view a section of F as a section of the ambient vector bundle E.

Definition 20.2. A ***k-frame*** of a C^∞ vector bundle $\pi\colon E \to M$ over an open set U in a manifold M is a collection of sections $s_1, \dots, s_k$ of E over U such that at every point p in U, the vectors $s_1(p), \dots, s_k(p)$ are linearly independent in the fiber E_p. In this terminology, a ***frame*** for a vector bundle of rank r is an r-frame.

Lemma 20.3. *Let $\pi\colon E \to M$ be a C^∞ vector bundle of rank r and let $p \in M$. Fix a positive integer $k \le r$. A C^∞ k-frame $s_1, \dots, s_k$ for E defined on a neighborhood U of p can be extended to a C^∞ r-frame for E on a possibly smaller neighborhood W of p.*

Proof. Replacing U by a smaller neighborhood of p if necessary, we may assume that there is a frame $e_1, \dots, e_r$ for E over U. On U each s_j is a linear combination $s_j = \sum a^i_j e_i$ with $a^i_j \in C^\infty(U)$. In matrix notation

$$[s_1 \ \cdots \ s_k] = [e_1 \ \cdots \ e_r][a^i_j].$$

The $r \times k$ matrix $a = [a^i_j]$ has rank k at every point in U. By renumbering $e_1, \ldots, e_r$ we may assume that the top $k \times k$ block a' of a is nonsingular at the point p. Since nonsingularity is an open condition there is a neighborhood W of p on which the $k \times k$ block a' is nonsingular pointwise.

We can extend the $r \times k$ matrix $a = \begin{bmatrix} a' \\ * \end{bmatrix}$ to a nonsingular $r \times r$ matrix b by adjoining an $r \times (r-k)$ matrix $\begin{bmatrix} 0 \\ I \end{bmatrix}$:

$$b = \begin{bmatrix} a' & 0 \\ * & I \end{bmatrix}.$$

Then

$$[s_1 \ \cdots \ s_k \ e_{k+1} \ \cdots \ e_r] = [e_1 \ \cdots \ e_r] b.$$

Since b is an $n \times n$ nonsingular matrix at every point of W, the sections $s_1, \ldots, s_k$, e_{k+1}, ..., e_r form a frame for E over W. □

20.2 Subbundle Criterion

A vector bundle is a locally trivial family of vector spaces over a base space. The following theorem gives a sufficient condition for a family of vector subspaces over the same base space to be locally trivial and therefore to be a subbundle.

Theorem 20.4. *Let $\pi \colon E \to M$ be a C^∞ vector bundle of rank r and $F := \coprod_{p \in M} F_p$ a subset of E such that for every p in M, the set F_p is a k-dimensional vector subspace of the fiber E_p. If for every p in M, there exist a neighborhood U of p and $m \geq k$ smooth sections $s_1, \ldots, s_m$ of E over U that span F_q at every point $q \in U$, then F is a C^∞ subbundle of E.*

Proof. By making a neighborhood W of p in U sufficiently small, one may ensure the triviality of E over W, so that there exists a local frame $e_1, \ldots, e_r$ for E over W. On W, each section s_j is a linear combination $s_j = \sum_{i=1}^r a^i_j e_i$ for an $r \times m$ matrix $a = [a^i_j]$ of C^∞ functions. In matrix notation

$$s := [s_1 \ \cdots \ s_m] = [e_1 \ \cdots \ e_r][a^i_j] = ea.$$

At every point of W the matrix a has rank k, because the columns of a are simply the column vectors of $s_1, \ldots, s_m$ relative to the basis $e_1, \ldots, e_r$ and $s_1, \ldots, s_m$ span a k-dimensional vector space pointwise. In particular, at the point p the matrix $a(p) = [a^i_j(p)]$ has rank k. It follows that $a(p)$ has a nonsingular $k \times k$ submatrix $a'(p)$. By renumbering $s_1, \ldots, s_m$ and $e_1, \ldots, e_r$, we may assume that $a'(p)$ is the left uppermost $k \times k$ submatrix of $a(p)$. Since the nonsingularity of a matrix is an open condition, a' is nonsingular on a neighborhood W' of p in W. At every point of W', since a' has rank k, so does the matrix $[s_1 \ \cdots \ s_k]$. This proves that $s_1, \ldots, s_k$ form a frame for F over W'.

By Lemma 20.3, $s_1,\dots,s_k$ can be extended to a C^∞ frame $t_1,\dots,t_k,\dots,t_r$ for E, with $t_i = s_i$ for $i = 1,\dots,k$, over a possibly smaller neighborhood W'' of p in W'. For any $v \in E|_{W''}$, let $v = \sum_{i=1}^r c^i(v)t_i(\pi(v))$. Then $F|_{W''}$ is defined by $c^{k+1} = 0, \dots, c^r = 0$ on $E|_{W''}$. This proves that F is a regular submanifold of E, because the C^∞ trivialization

$$\begin{aligned}\phi_{W''}\colon E|_{W''} &\xrightarrow{\sim} W''\times\mathbb{R}^r,\\ v &\mapsto \big(\pi(v), c^1(v),\dots,c^r(v)\big),\end{aligned}$$

induces a bijection $\psi_{W''}\colon F|_{W''} \to W''\times\mathbb{R}^k$. (The c^i's are C^∞ by Problem 7.5.) As the restriction of a C^∞ map, $\psi_{W''}$ is C^∞. Its inverse,

$$\psi_{W''}^{-1}\colon (p,c^1,\dots,c^k) \mapsto \sum_{i=1}^k c^i t_i(p) = \sum_{i=1}^k c^i s_i(p),$$

is C^∞. Hence, F is a C^∞ vector bundle of rank k. The inclusion $F \hookrightarrow E$, locally given by

$$\begin{aligned}W''\times\mathbb{R}^k &\to W''\times\mathbb{R}^r,\\ (p,c^1,\dots,c^k) &\mapsto (p,c^1,\dots,c^k,0,\dots,0)\end{aligned}$$

is clearly a bundle map. Thus, F is a smooth rank-k subbundle of E. □

20.3 Quotient Bundles

Suppose F is a C^∞ subbundle of a C^∞ vector bundle $\pi\colon E \to M$. At each point p in M, the fiber F_p is a vector subspace of E_p and so the quotient space $Q_p := E_p/F_p$ is defined. Let

$$Q := \coprod_{p\in M} Q_p = \coprod_{p\in M} (E_p/F_p),$$

and give Q the quotient topology as a quotient space of E. Let $\rho\colon E \to Q$ be the quotient map. The projection $\pi\colon E \to M$ then induces a map $\pi_Q\colon Q \to M$ as in the commutative diagram

$$\begin{array}{ccccc} F & \subset & E & \xrightarrow{\ \rho\ } & Q \\ & & {\scriptstyle\pi}\downarrow & \swarrow{\scriptstyle\pi_Q} & \\ & & M. & & \end{array}$$

Since F is locally trivial, every point p in M has a coordinate neighborhood over which one can find a C^∞ frame $s_1,\dots,s_k$ for F. By Lemma 20.3, $s_1,\dots,s_k$ can be extended to a C^∞ frame $s_1,\dots,s_k,s_{k+1},\dots,s_r$ for E over a possibly smaller neighborhood W of p. A point v of $E|_W$ is uniquely a linear combination

$$v = \sum c^i(v) s_i\big(\pi(v)\big).$$

The functions c^i on $E|_W$ are coordinate functions and hence are C^∞.

Let $\bar{s}_{k+1}, \dots, \bar{s}_r \colon W \to Q$ be the sections $s_{k+1}, \dots, s_r$ followed by the projection $E \to Q$. Then $(\bar{s}_{k+1}, \dots, \bar{s}_r)$ is a continuous frame for Q over W, and every element $\bar{v} \in Q|_W := \pi_Q^{-1}(W)$ can be written uniquely as a linear combination

$$\bar{v} = \sum \bar{c}^i(\bar{v}) \bar{s}_i\big(\pi_Q(\bar{v})\big).$$

This gives rise to a bijection

$$\begin{aligned} \phi_W \colon Q|_W &\to W \times \mathbb{R}^{r-k}, \\ \bar{v} &\mapsto \Big(p, \bar{c}^{k+1}(\bar{v}), \dots, \bar{c}^r(\bar{v})\Big), \quad p = \pi_Q(\bar{v}). \end{aligned}$$

The maps $\bar{c}^i$ are continuous on $Q|_W$ because their lifts $\bar{c}^i \circ \rho = c^i$ to $E|_W$ are continuous. Thus, ϕ_W is a continuous map. Since

$$\phi_W^{-1}(p, \bar{c}^{k+1}, \dots, \bar{c}^r) = \sum \bar{c}^i \bar{s}_i(p)$$

is clearly continuous, ϕ_W is a homeomorphism. Using the homeomorphisms ϕ_W, one can give Q a manifold structure as well as a C^∞ vector bundle structure over M. With this vector bundle structure, $\pi_Q \colon Q \to M$ is called the ***quotient bundle*** of E by F.

20.4 The Pullback Bundle

If $\pi \colon E \to M$ is a C^∞ vector bundle over a manifold M and $f \colon N \to M$ is a C^∞ map, then there is a C^∞ vector bundle f^*E over N, called the ***pullback*** of E by f, with the property that every bundle map covering f factors through the pullback bundle f^*E (see Proposition 20.8).

The total space of the ***pullback bundle*** of E by f is defined to be the set

$$f^*E = \{(n, e) \in N \times E \mid f(n) = \pi(e)\},$$

endowed with the subspace topology. The projections to the two factors,

$$\begin{aligned} \eta \colon f^*E &\to N, & \zeta \colon f^*E &\to E, \\ \eta(n, e) &= n, & \zeta(n, e) &= e, \end{aligned}$$

fit into a commutative diagram

$$\begin{array}{ccc} f^*E & \xrightarrow{\ \zeta\ } & E \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow[\ f\ ]{} & M. \end{array} \tag{20.1}$$

We will show that $\eta \colon f^*E \to N$ is a vector bundle. First, we show that the pullback of a product bundle is a product bundle.

Proposition 20.5. *If $f\colon N \to M$ is a C^∞ map of manifolds and $\pi\colon E = M \times V \to M$ is a product bundle, then the projection $\eta\colon f^*E \to N$ is isomorphic to the product bundle $N \times V \to N$.*

Proof. As a set,

$$\begin{aligned} f^*E &= \{\big(n,(m,v)\big) \in N \times (M \times V) \mid f(n) = \pi(m,v) = m\} \\ &= \{\big(n,(f(n),v)\big) \in N \times (M \times V)\}. \end{aligned}$$

The map

$$\begin{gathered} \sigma\colon f^*E \to N \times V, \\ \big(n,(f(n),v)\big) \mapsto (n,v) \end{gathered}$$

with inverse $(n,v) \mapsto \big(n,(f(n),v)\big)$ is a fiber-preserving homeomorphism. It gives $\eta\colon f^*E \to N$ the structure of a C^∞ vector bundle over N. □

Theorem 20.6. *Let $\pi\colon E \to M$ be a C^∞ vector bundle with fiber V and $f\colon N \to M$ a C^∞ map. The projection $\eta\colon f^*E \to N$ can be given the structure of a C^∞ vector bundle with fiber V.*

For any map $\pi\colon E \to M$ and open set $U \subset M$, recall that the ***restriction*** of E to U is denoted $E|_U := \pi^{-1}(U) \to U$.

Proof. Since E is locally a product $U \times V \to U$, by Proposition 20.5, the pullback f^*E is locally the product $f^{-1}(U) \times V \to f^{-1}(U)$. □

Lemma. *Suppose $\pi\colon E \to M$ is a C^∞ vector bundle with fiber V and $f\colon N \to M$ is a C^∞ map. Let U be an open subset of M. Then*

$$(f^*E)|_{f^{-1}(U)} = f^*(E|_U).$$

Proof. By definition,

$$\begin{aligned} (f^*E)|_{f^{-1}(U)} &= \{(n,e) \in N \times E \mid n \in f^{-1}(U), f(n) = \pi(e)\} \\ &= \{(n,e) \in f^{-1}(U) \times E \mid f(n) = \pi(e)\}. \\ f^*(E|_U) &= \{(n,e) \in f^{-1}(U) \times E|_U \mid f(n) = \pi(e)\}. \end{aligned}$$

Comparing the two, we see that they are equal. □

Proposition 20.7. *Suppose $\pi\colon E \to M$ is a C^∞ vector bundle with fiber V and trivializing open cover $\{U_\alpha\}$ and $f\colon N \to M$ is a C^∞ map. If $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(V)$ is the transition function for E over $U_\alpha \cap U_\beta$, then $f^*g_{\alpha\beta}$ is the transition function for f^*E over $f^{-1}(U_\alpha \cap U_\beta)$.*

Proof. Suppose $\phi_\alpha \colon E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times V$ is the trivialization for E over U_α, with

$$\phi_\alpha(e) = \big(\pi(e), \bar{\phi}_\alpha(e)\big).$$

Then

$$f^*(E|_{U_\alpha}) = (f^*E)|_{f^{-1}(U_\alpha} \xrightarrow{\sim} f^{-1}(U) \times V$$

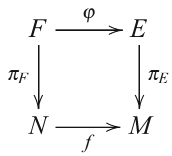
is given by

$$(n,e) \mapsto \big(n, \bar{\phi}_\alpha(e)\big).$$

So the transition function for f^*E over $f^{-1}(U_\alpha) \cap f^{-1}(U_\beta)$ is

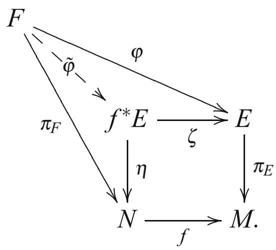
$$(\bar{\phi}_\alpha \circ \bar{\phi}_\beta^{-1})\big(f(n)\big) = (f^* g_{\alpha\beta})(n). \qquad \square$$

Proposition 20.8. *Suppose $\pi_F \colon F \to N$ and $\pi_E \colon E \to M$ are vector bundles and $\varphi \colon F \to E$ is a bundle map that covers $f \colon N \to M$, i.e., the diagram*



(20.2)

commutes. Then there is a unique bundle map $\tilde{\varphi} \colon F \to f^*E$ over N that makes the following diagram commute:



(20.3)

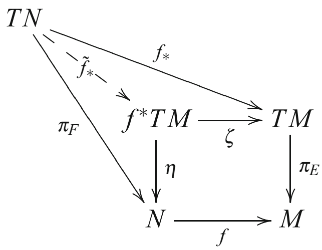
Proof. For all $q \in F$, the commutativity of the diagram (20.3) forces $\tilde{\varphi}(q) = \big(\pi_F(q), \varphi(q)\big)$. This shows that $\tilde{\varphi}$ is unique if it exists. Because φ covers f, $f(\pi_F(q)) = \pi_E\big(\varphi(q)\big)$. Hence, $\big(\pi_F(q), \varphi(q)\big) \in f^*E$. So the map $\tilde{\varphi}$ as defined above indeed exists. It is $\mathbb{R}$-linear on each fiber because φ is. It is continuous because π_F and φ are assumed continuous. $\square$

This proposition shows that given f and E, the commutative diagram (20.1) of the pullback bundle f^*E is a final object among all commutative diagrams of the form (20.2).

20.5 Examples of the Pullback Bundle

The pullback construction for vector bundles is indispensable in differential geometry. In this section we give two examples. The first uses the pullback bundle to convert a bundle map over two different base manifolds to a bundle map over a single manifold. This is sometimes desirable in order to obtain an exact sequence of vector bundles over a manifold (see (27.5)). The second example uses the pullback bundle to clarify the notion of a vector field along a curve in a manifold, which we encountered in Sections 4.3 and 13.1.

Example 20.9 (*The differential of a map*). If $f\colon N \to M$ is a C^∞ map of manifolds, its differentials $f_{*,p}\colon T_pN \to T_{f(p)}M$ at all points $p \in N$ piece together to give a bundle map $f_*\colon TN \to TM$ of tangent bundles. By Proposition 20.8, the bundle map f_* induces a unique bundle map $\widetilde{f_*}\colon TN \to f^*TM$ over N that makes the diagram



commutative. The map $\widetilde{f_*}\colon TN \to f^*TM$ is given by

$$X_p \in T_pN \mapsto (p, f_{*,p}X_p \in T_{f(p)}M).$$

Conversely, f_* can be obtained from $\widetilde{f_*}$ as $f_* = \zeta \circ \widetilde{f_*}$. In this way the bundle map f_* over two base manifolds is converted to a bundle map $\widetilde{f_*}$ over the single manifold N.

Example 20.10 (*Vector fields along a curve*). If $c\colon I \to M$ is a smooth map from an open interval $I \subset \mathbb{R}$ into a manifold M, then the pullback c^*TM of the tangent bundle TM is a vector bundle over I. A section of the pullback bundle c^*TM assigns to each $t \in I$ an element of the fiber $(c^*TM)_t \simeq T_{c(t)}M$, i.e., a tangent vector to M at $c(t)$. In other words, a section of c^*TM is precisely a vector field along the curve $c(t)$ in M defined in Section 4.3.

Exercise 20.11. Show that a vector field along a curve c in M is smooth if and only if the corresponding section of c^*TM is smooth.

Thus, we can identify the space of smooth vector fields along $c(t)$ with the space of smooth sections of the pullback bundle c^*TM:

$$\Gamma(TM|_{c(t)}) = \Gamma(c^*TM).$$

20.6 The Direct Sum of Vector Bundles

Suppose $\pi\colon E \to M$ and $\pi'\colon E' \to M$ are two smooth vector bundles of ranks r and r', respectively, over a manifold M. For $p \in M$, denote by E_p the fiber $\pi^{-1}(p)$ of E over p, and by E'_p the fiber $(\pi')^{-1}(p)$ of E' over p. Define the ***direct sum*** $E \oplus E'$ as a set to be the disjoint union

$$E \oplus E' := \coprod_{p\in M} E_p \oplus E'_p := \bigcup_{p\in M} \{p\} \times (E_p \oplus E'_p). \tag{20.4}$$

Let $\rho\colon E \oplus E' \to M$ be the projection $\rho\big(p,(e,e')\big) = p$. We will now need to put a topology and a manifold structure on $E \oplus E'$ so that $\rho\colon E \oplus E' \to M$ becomes a C^∞ vector bundle with fibers $E_p \oplus E'_p$.

Lemma 20.12. *If $\{U_\alpha\}_{\alpha\in \mathrm{A}}$ and $\{V_\beta\}_{\beta\in \mathrm{B}}$ are open covers of a topological space M, then $\{U_\alpha \cap V_\beta\}_{(\alpha,\beta)\in \mathrm{A}\times\mathrm{B}}$ is again an open cover of M.*

Proof. Since $\{V_\beta\}$ is an open cover of M, for each $\alpha \in A$,

$$U_\alpha = U_\alpha \cap M = U_\alpha \cap \left(\bigcup_\beta V_\beta\right) = \bigcup_\beta (U_\alpha \cap V_\beta).$$

Thus

$$M = \bigcup_\alpha U_\alpha = \bigcup_{\alpha,\beta} (U_\alpha \cap V_\beta). \qquad \square$$

If $\{U_\alpha\}_{\alpha\in\mathrm{A}}$ and $\{V'_\beta\}_{\beta\in\mathrm{B}}$ are trivializing open covers for E and E', respectively, then by Lemma 20.12 $\{U_\alpha \cap V'_\beta\}_{\alpha\in\mathrm{A},\beta\in\mathrm{B}}$ is an open cover of M that simultaneously trivializes both E and E'.

Choose a *coordinate open cover* $\mathfrak{U} = \{U_\alpha\}_{\alpha\in\mathrm{A}}$ of M that simultaneously trivializes both E and E', with trivializations

$$\psi_\alpha\colon\ E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^r \quad \text{and} \quad \psi'_\alpha\colon\ E'|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^{r'}.$$

By adding all finite intersections $U_{\alpha_1} \cap \cdots \cap U_{\alpha_i}$ to the open cover $\mathfrak{U}$, we may assume that if $U_\alpha, U_\beta \in \mathfrak{U}$, then $U_\alpha \cap U_\beta \in \mathfrak{U}$. For each $p \in U_\alpha$, there are linear isomorphisms

$$\psi_{\alpha,p}\colon E_p \xrightarrow{\sim} \{p\} \times \mathbb{R}^r \quad \text{and} \quad \psi'_{\alpha,p}\colon E'_p \xrightarrow{\sim} \{p\} \times \mathbb{R}^{r'}.$$

By the functorial property of the direct sum, there is an induced linear isomorphism

$$\phi_{\alpha,p} := \psi_{\alpha,p} \oplus \psi'_{\alpha,p}\colon E_p \oplus E'_p \xrightarrow{\sim} \{p\} \times \mathbb{R}^{r+r'},$$

and hence a bijection

$$\phi_\alpha\colon \coprod_{p\in U_\alpha} E_p \oplus E'_p = \rho^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^{r+r'}$$

that is a linear isomorphism on each fiber.

To simplify the notation we will sometimes write $U_{\alpha\beta} = U_\alpha \cap U_\beta$. For $\alpha, \beta \in \mathrm{A}$, the transition function

$$\phi_\alpha \circ \phi_\beta^{-1} \colon U_{\alpha\beta} \times \mathbb{R}^{r+r'} \to \rho^{-1}(U_{\alpha\beta}) \to U_{\alpha\beta} \times \mathbb{R}^{r+r'}$$

restricts on each fiber to

$$\phi_{\alpha,p} \circ \phi_{\beta,p}^{-1} \colon \{p\} \times \mathbb{R}^{r+r'} \to E_p \oplus E'_p \to \{p\} \times \mathbb{R}^{r+r'}, \qquad p \in U_{\alpha\beta}.$$

If we set $g_{\alpha\beta}(p) = \psi_{\alpha,p} \circ \psi_{\beta,p}^{-1}$ and $g'_{\alpha\beta}(p) = \psi'_{\alpha,p} \circ (\psi'_{\beta,p})^{-1}$, then

$$\begin{aligned}(\phi_\alpha \circ \phi_\beta^{-1})\big(p,(v,v')\big) &= \Big(p, (\psi_{\alpha,p} \circ \psi_{\beta,p}^{-1})v, (\psi'_{\alpha,p} \circ (\psi'_{\beta,p})^{-1})v'\Big) \\ &= \Big(p, \big(g_{\alpha\beta}(p)v, g'_{\alpha\beta}(p)v'\big)\Big).\end{aligned}$$

This shows that $\phi_\alpha \circ \phi_\beta^{-1}$ is a continuous map, because $g_{\alpha\beta} \colon U_{\alpha\beta} \to \mathrm{GL}(r,\mathbb{R})$ and $g'_{\alpha\beta} \colon U_{\alpha\beta} \to \mathrm{GL}(r',\mathbb{R})$ are both continuous. As its inverse $\phi_\beta \circ \phi_\alpha^{-1}$ is also continuous, $\phi_\alpha \circ \phi_\beta^{-1}$ is a homeomorphism.

Since $U_\alpha \times \mathbb{R}^{r+r'}$ has a topology, we can use the trivialization ϕ_α to define a topology on $\rho^{-1}(U_\alpha)$ so that ϕ_α becomes a homeomorphism. We now have a collection $\{\rho^{-1}(U_\alpha)\}_{\alpha \in \mathrm{A}}$ of subsets of $E \oplus E'$ such that

(i) $E \oplus E' = \bigcup_{\alpha \in \mathrm{A}} \rho^{-1}(U_\alpha)$;
(ii) for any $\alpha, \beta \in \mathrm{A}$,

$$\rho^{-1}(U_\alpha) \cap \rho^{-1}(U_\beta) = \rho^{-1}(U_{\alpha\beta}),$$

which is again in the collection;
(iii) for each $\alpha \in \mathrm{A}$, there is a bijection

$$\phi_\alpha \colon \rho^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{r+r'};$$

(iv) for each pair $\alpha, \beta \in \mathrm{A}$, the map

$$\phi_\alpha \circ \phi_\beta^{-1} \colon \phi_\beta\left(\rho^{-1}(U_{\alpha\beta})\right) \to \phi_\alpha\left(\rho^{-1}(U_{\alpha\beta})\right)$$

is a homeomorphism.

We formalize this situation in a topological lemma.

Lemma 20.13. *Let $\mathcal{C} = \{S_\alpha\}_{\alpha \in \mathrm{A}}$ be a collection of subsets of a set S such that*

(i) *their union is S;*
(ii) *$\mathcal{C}$ is closed under finite intersections;*
(iii) *for each $\alpha \in \mathrm{A}$, there is a bijection*

$$\phi_\alpha \colon S_\alpha \to Y_\alpha,$$

where Y_α is a topological space;

(iv) *for each pair* $\alpha, \beta \in \mathrm{A}$, $\phi_\alpha(S_{\alpha\beta})$ *is open in* Y_α, *where* $S_{\alpha\beta} = S_\alpha \cap S_\beta$, *and the map*

$$\phi_\alpha \circ \phi_\beta^{-1} \colon \phi_\beta(S_{\alpha\beta}) \to \phi_\alpha(S_{\alpha\beta})$$

is a homeomorphism.

Then there is a unique topology on S *such that for each* $\alpha \in \mathrm{A}$, *the subset* S_α *is open and the bijection* $\phi_\alpha \colon S_\alpha \to Y_\alpha$ *is a homeomorphism.*

The proof of this lemma is left as an exercise (Problem 20.1). Applying the lemma to the collection $\{\rho^{-1}(U_\alpha)\}_{\alpha\in\mathrm{A}}$ of subsets of $E \oplus E'$, we obtain a topology on $E \oplus E'$ in which each $\rho^{-1}(U_\alpha)$ is open and each

$$\phi_\alpha \colon \rho^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{r+r'} \tag{20.5}$$

is a homeomorphism. Since the sets $U_\alpha \times \mathbb{R}^{r+r'}$ are homeomorphic to open subsets of $\mathbb{R}^{n+r+r'}$ and the transition functions $\phi_\alpha \circ \phi_\beta^{-1}$ are all C^∞, $\{(\rho^{-1}(U_\alpha), \phi_\alpha)\}$ is a C^∞ atlas on $E \oplus E'$. Thus, $E \oplus E'$ is a C^∞ manifold. Moreover, the trivializations (20.5) show that $\rho \colon E \oplus E' \to M$ is a C^∞ vector bundle of rank $r + r'$.

20.7 Other Operations on Vector Bundles

The construction of the preceding section can be applied to the disjoint union $\coprod_{p\in M} E_p \otimes E'_p$ of the tensor product of fibers of E and E'. This produces a C^∞ vector bundle $E \otimes E'$ of rank rr' over M, whose fiber above $p \in M$ is the tensor product $E_p \otimes E'_p$.

More generally, let $\mathcal{V}$ be the category whose objects are finite-dimensional real vector spaces and whose morphisms are *isomorphisms*, not merely linear maps, of vector spaces. In this category, if two vector spaces have different dimensions, then the set of morphisms between them is the empty set. Denote by $\mathcal{V} \times \mathcal{V}$ the category whose objects are pairs of finite-dimensional vector spaces and whose morphisms are pairs of isomorphisms of vector spaces. Let $\mathcal{T} \colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ be a covariant functor that associates to a pair of finite-dimensional vector spaces (V, W) another finite-dimensional vector space $\mathcal{T}(V, W)$, and to a pair of isomorphisms $(f \colon V \to V', g \colon W \to W')$ an isomorphism $\mathcal{T}(f, g) \colon \mathcal{T}(V, W) \to \mathcal{T}(V', W')$.

If V and V' are finite-dimensional vector spaces of the same dimension n, let $\mathrm{Iso}(V, V')$ be the set of all isomorphisms from V to V'. With respect to fixed bases for V and V', elements of $\mathrm{Iso}(V, V')$ are represented by nonsingular $n \times n$ matrices. Hence, $\mathrm{Iso}(V, V')$ is bijective with $\mathrm{GL}(n, \mathbb{R})$, and therefore has the structure of a manifold as an open subset of $\mathbb{R}^{n\times n}$. The functor $\mathcal{T} \colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is said to be ***smooth*** or C^∞ if for all finite-dimensional vector spaces V, V', W, W' with $\dim V = \dim V'$ and $\dim W = \dim W'$, the map

$$\mathcal{T} \colon \mathrm{Iso}(V, V') \times \mathrm{Iso}(W, W') \to \mathrm{Iso}\left(\mathcal{T}(V, W), \mathcal{T}(V', W')\right),$$
$$(f, g) \mapsto \mathcal{T}(f, g),$$

is smooth.

Example. Let $\mathcal{T}(V,W) = V \oplus W$. This is a smooth functor because if $f\colon V \to V'$ and $g\colon W \to W'$ are isomorphisms represented by matrices M_f and M_g relative to some bases, then $f \oplus g\colon V \oplus W \to V' \oplus W'$ is represented by the matrix

$$\begin{bmatrix} M_f & 0 \\ 0 & M_g \end{bmatrix}$$

relative to the same bases.

Mimicking the construction of the direct sum of two vector bundles, we obtain the following result.

Proposition 20.14. *If $\mathcal{T}\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is a C^∞ covariant functor, then for any two C^∞ vector bundles E and F over a manifold M, there is a C^∞ vector bundle $\mathcal{T}(E,F)$ over M whose fiber at $p \in M$ is $\mathcal{T}(E_p, F_p)$.*

The same construction applies if $\mathcal{T}$ has any number of arguments. If a functor is contravariant, it must be turned into a covariant functor first to apply this construction. For example, consider the dual functor applied to a vector bundle $E \to M$. Because of its contravariance, it associates to a trivialization

$$\psi_p\colon E_p \to \{p\} \times \mathbb{R}^r$$

the map

$$\psi_p^\vee\colon \{p\} \times (\mathbb{R}^r)^\vee \to E_p^\vee,$$

which has the wrong direction. We need to take the inverse of $\psi_p^\vee$ to get a trivialization that goes in the right direction,

$$(\psi_p^\vee)^{-1}\colon E_p^\vee \to \{p\} \times (\mathbb{R}^r)^\vee.$$

To construct the dual bundle, the functor $\mathcal{T}\colon \mathcal{V} \to \mathcal{V}$ associates to every finite-dimensional vector space V its dual space $V^\vee$, and to every isomorphism $f\colon V \to W$ the isomorphism $(f^\vee)^{-1}\colon V^\vee \to W^\vee$.

In the category $\mathcal{V}$ the morphisms are all isomorphisms precisely so that one can reverse the direction of a map and make a contravariant functor covariant. In this way, starting from C^∞ vector bundles E and F over M, one can construct C^∞ vector bundles $E \oplus F$, $E \otimes F$, $\bigwedge^k E$, $E^\vee$, and $\operatorname{Hom}(E,F) \simeq E^\vee \otimes F$ over M.

In Section 10.4 we defined a Riemannian metric on a vector bundle $E \to M$ as a C^∞ assignment of an inner product $\langle\ ,\ \rangle_p$ on the fiber E_p to each point p in M. Using the multilinear algebra and vector bundle theory from the preceding sections, we can be more precise about the kind of object a Riemannian metric is.

An inner product on a real vector space V is first of all a bilinear map: $V \times V \to \mathbb{R}$. By the universal mapping property, it can be viewed as a linear map: $V \otimes V \to \mathbb{R}$, or an element of $\operatorname{Hom}(V \otimes V, \mathbb{R})$. By Proposition 18.15,

$$\operatorname{Hom}(V \otimes V, \mathbb{R}) \simeq (V \otimes V)^\vee \simeq V^\vee \otimes V^\vee.$$

Thus, an inner product on V is an element of $V^\vee \otimes V^\vee$ which is positive-definite and symmetric as bilinear maps.

A Riemannian metric on the vector bundle $E \to M$ associates to each point $p \in M$ an element of $E_p^\vee \otimes E_p^\vee$; in other words, it is a C^∞ section of $E^\vee \otimes E^\vee$ satisfying the positivity and symmetry conditions.

Problems

20.1.* Topology of a union
Prove Lemma 20.13.

20.2. Exterior power of a vector bundle
For a smooth vector bundle E over a manifold M, give the details of the construction of $\bigwedge^k E$ that show it to be a smooth vector bundle over M whose fiber at $p \in M$ is $\bigwedge^k (E_p)$.

20.3. Tensor product of two vector bundles
For two smooth vector bundles E and F over a manifold M, give the details of the construction of $E \otimes F$ that show it to be a smooth vector bundle over M whose fiber at $p \in M$ is $E_p \otimes F_p$.

§21 Vector-Valued Forms

A differential k-form α on a manifold M assigns to each point $p \in M$ an alternating k-linear map

$$\alpha_p \colon T_pM \times \cdots \times T_pM \to \mathbb{R}.$$

If instead of $\mathbb{R}$, the target space of the map α_p is a vector space V, then α is called a ***vector-valued form***. Vector-valued forms arise naturally in differential geometry, for example, when one wants to define connections and curvature for a principal bundle as we do in Chapter 6. In preparation, we gather together in this section some basic constructions and properties of vector-valued forms.

Even more generally, the target space of α_p may be the fiber E_p of a vector bundle E over M. In this case, α is a differential form with values in a vector bundle. We have already encountered such an object: from Section 10.3, the curvature of a connection on a vector bundle $E \to M$ can be seen as a 2-form with values in the endomorphism bundle $\operatorname{End}(E)$.

Throughout this section, V, W, and Z denote finite-dimensional real vector spaces.

21.1 Vector-Valued Forms as Sections of a Vector Bundle

Let M be a manifold and p a point in M. By the universal mapping property for an exterior power, there is a one-to-one correspondence between alternating k-linear maps on the tangent space T_pM and linear maps from $\bigwedge^k T_pM$ to $\mathbb{R}$. If $A_k(T_pM)$ is the space of alternating k-linear functions on the tangent space T_pM, then by (19.3) and Theorem 19.13, there are canonical isomorphisms

$$A_k(T_pM) \simeq \left(\bigwedge^k T_pM\right)^\vee \simeq \bigwedge^k (T_p^*M).$$

Thus, a C^∞ k-form on M may be viewed as a C^∞ section of the vector bundle $\bigwedge^k T^*M$, and so the vector space of C^∞ k-forms on M is

$$\Omega^k(M) = \Gamma\left(\bigwedge^k T^*M\right).$$

It is a simple matter to generalize the usual calculus of differential forms on a manifold to differential forms with values in a finite-dimensional vector space V, or ***V-valued forms*** for short.

Let T^k be the Cartesian product of k copies of a vector space T. A ***V-valued k-covector*** on T is an alternating k-linear function $f \colon T^k \to V$. By the universal mapping property of the exterior power $\bigwedge^k T$ (Theorem 19.6), to each V-valued k-covector $f \colon T^k \to V$ there corresponds a unique linear map $\tilde{f} \colon \bigwedge^k T \to V$ such that the diagram

$$\begin{array}{ccc} \bigwedge^k T & & \\ \uparrow{\scriptstyle\wedge} & \searrow^{\tilde{f}} & \\ T^k & \xrightarrow[f]{} & V \end{array} \tag{21.1}$$

commutes, and conversely. We denote the vector space of V-valued k-covectors on T by $A_k(T,V)$. The one-to-one correspondence $f \mapsto \tilde{f}$ in (21.1) induces a linear isomorphism

$$A_k(T,V) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{R}}\left(\bigwedge^k T, V\right).$$

By two standard isomorphisms of multilinear algebra (Proposition 18.14 and Theorem 19.13),

$$\operatorname{Hom}_{\mathbb{R}}\left(\bigwedge^k T, V\right) \simeq \left(\bigwedge^k T\right)^\vee \otimes V \simeq \left(\bigwedge^k T^\vee\right) \otimes V.$$

It is customary to write $T^\vee$ as T^*. Thus, a V-valued k-covector on T is an element of the vector space $(\bigwedge^k T^*) \otimes V$.

A ***V-valued k-form*** on a manifold M is a function that assigns to each point $p \in M$ a V-valued k-covector on the tangent space T_pM; equivalently, it assigns to each $p \in M$ an element of $(\bigwedge^k T_p^*M) \otimes V$. If E is a vector bundle over M and V is a vector space, the notation $E \otimes V$ will mean the tensor product of the vector bundle E with the product bundle $M \times V \to M$. Then a V-valued k-form is a section of the vector bundle $(\bigwedge^k T^*M) \otimes V$. We denote the space of smooth V-valued k-forms on M by

$$\Omega^k(M,V) := \Gamma\left(\left(\bigwedge^k T^*M\right) \otimes V\right).$$

Let $v_1, \ldots, v_n$ be a basis for V, and α a V-valued k-form on a manifold M. For any point $p \in M$ and tangent vectors $u_1, \ldots, u_k \in T_pM$, since $\alpha_p(u_1, \ldots, u_k)$ is an element of V, it is uniquely a linear combination of $v_1, \ldots, v_n$ with real coefficients depending on p and $u_1, \ldots, u_k$. We denote the coefficient of v_i by $\alpha_p^i(u_1, \ldots, u_k)$. Then

$$\alpha_p(u_1, \ldots, u_k) = \sum_i \alpha_p^i(u_1, \ldots, u_k) v_i. \tag{21.2}$$

Because α_p is alternating and k-linear, so is $\alpha_p^i(u_1, \ldots, u_k)$ for each i. Thus, α_p^i is a k-covector with values in $\mathbb{R}$. We can rewrite (21.2) as

$$\alpha_p(u_1, \ldots, u_k) = \sum_i (\alpha_p^i \otimes v_i)(u_1, \ldots, u_k).$$

(This notation is consistent with that of the tensor product of a k-linear function with 0-linear function, i.e., with a constant.) As p varies in M, we see that every V-valued k-form α on M is a linear combination

$$\alpha = \sum \alpha^i \otimes v_i,$$

where the α^i are ordinary k-forms on M. We usually omit the tensor product sign and write more simply $\alpha = \sum \alpha^i v_i$. A k-form is also called a form of ***degree*** k. A form is ***homogeneous*** if it is a sum of forms all of the same degree. The V-valued k-form α is said to be ***smooth*** if the coefficients α^i are smooth for all $i = 1, \dots, n$. It is immediate that this definition of smoothness is independent of the choice of basis.

21.2 Products of Vector-Valued Forms

Let V, W, Z be finite-dimensional vector spaces and $\mu\colon V \times W \to Z$ a bilinear map. One can define a product of a V-valued covector and a W-valued covector on a vector space T,

$$A_k(T,V) \times A_\ell(T,W) \to A_{k+\ell}(T,Z),$$

by the same formula as the wedge product of two scalar forms, with the bilinear map μ replacing the multiplication of real numbers:

$$\begin{aligned}(\alpha \cdot \beta)(t_1, \dots, t_{k+\ell}) \\ = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma) \mu\big(\alpha(t_{\sigma(1)}, \dots, t_{\sigma(k)}), \beta(t_{\sigma(k+1)}, \dots, t_{\sigma(k+\ell)})\big) \qquad (21.3)\end{aligned}$$

for $\alpha \in A_k(T,V)$ and $\beta \in A_\ell(T,W)$.

This formula generalizes the wedge product of scalar covectors. The same proofs as in the scalar case [21, Section 3.7, pp. 26–27] show that

(i) $\alpha \cdot \beta$ is alternating and multilinear in its argument, and hence is a $(k+\ell)$-covector on T;
(ii) instead of summing over all permutations in $S_{k+\ell}$, one may sum over (k,ℓ)-shuffles:

$$\begin{aligned}(\alpha \cdot \beta)(t_1, \dots, t_{k+\ell}) \\ = \sum_{\substack{(k,\ell)\text{-shuffles} \\ \sigma}} (\operatorname{sgn} \sigma) \mu\big(\alpha(t_{\sigma(1)}, \dots, t_{\sigma(k)}), \beta(t_{\sigma(k+1)}, \dots, t_{\sigma(k+\ell)})\big). \qquad (21.4)\end{aligned}$$

(By Definition 19.14, a (k,ℓ)-shuffle is a permutation σ in $S_{k+\ell}$ such that

$$\sigma(1) < \cdots < \sigma(k) \text{ and } \sigma(k+1) < \cdots < \sigma(k+\ell).)$$

This $\alpha \cdot \beta$ is a product in the sense of being $\mathbb{R}$-linear in α and in β. It is neither graded-commutative nor associative, even when $V = W = Z$.

When applied pointwise to a manifold, with $T = T_pM$, the product (21.3) gives rise to a bilinear map of vector-valued forms

$$\Omega^k(M,V) \times \Omega^\ell(M,W) \to \Omega^{k+\ell}(M,Z).$$

Proposition 21.1. *Let v_i be vectors in a vector space V and let w_j be vectors in a vector space W. Suppose $\mu\colon V\times W\to Z$ is a bilinear map, and $\alpha=\sum\alpha^i v_i\in\Omega^k(M,V)$ and $\beta=\sum\beta^j w_j\in\Omega^\ell(M,W)$ are vector-valued forms, where α^i and β^j are $\mathbb{R}$-valued forms on a manifold M. Then*

$$\alpha\cdot\beta=\sum_{i,j}(\alpha^i\wedge\beta^j)\mu(v_i,w_j)\in\Omega^{k+\ell}(M,Z).$$

Proof. Fix $p\in M$ and $u_1,\dots,u_{k+\ell}\in T_pM$. Then

$$\begin{aligned}
&(\alpha\cdot\beta)_p(u_1,\dots,u_{k+\ell})\\
&=\frac{1}{k!\ell!}\sum_{\sigma\in S_{k+\ell}}(\operatorname{sgn}\sigma)\mu\left(\alpha_p(u_{\sigma(1)},\dots,u_{\sigma(k)}),\beta_p(u_{\sigma(k+1)},\dots,u_{\sigma(k+\ell)})\right)\\
&=\frac{1}{k!\ell!}\sum_{\sigma\in S_{k+\ell}}\\
&\quad(\operatorname{sgn}\sigma)\mu\Big(\sum_i\alpha_p^i\left(u_{\sigma(1)},\dots,u_{\sigma(k)}\right)v_i,\sum_j\beta_p^j\left(u_{\sigma(k+1)},\dots,u_{\sigma(k+\ell)}\right)w_j\Big)\\
&=\sum_{i,j}\frac{1}{k!\ell!}\\
&\quad\sum_{\sigma\in S_{k+\ell}}(\operatorname{sgn}\sigma)\alpha_p^i\left(u_{\sigma(1)},\dots,u_{\sigma(k)}\right)\beta_p^j\left(u_{\sigma(k+1)},\dots,u_{\sigma(k+\ell)}\right)\mu(v_i,w_j)\\
&=\sum_{i,j}(\alpha^i\wedge\beta^j)_p(u_1,\dots,u_{k+\ell})\mu(v_i,w_j).
\end{aligned}$$

To show that $\alpha\wedge\beta$ is smooth, we may take $\{v_i\}$ and $\{w_j\}$ to be bases for V and W, respectively. Then α^i and β^j are smooth by definition. Let $\{z_k\}$ be a basis for Z and suppose

$$\mu(v_i,w_j)=\sum_k c_{ij}^k z_k.$$

By the formula just proven,

$$\alpha\cdot\beta=\sum_k\Big(\sum_{i,j}(\alpha^i\wedge\beta^j)c_{ij}^k\Big)z_k,$$

which shows that $\alpha\cdot\beta$ is smooth. □

Example 21.2. An $m\times n$ matrix-valued form is an element of $\Omega^*(M,\mathbb{R}^{m\times n})$. Let $e_{ij}\in\mathbb{R}^{m\times p}$ be the matrix with 1 in its (i,j)-entry and 0 in all other entries, and let $\bar{e}_{ij}\in\mathbb{R}^{p\times n}$ and $\tilde{e}_{ij}\in\mathbb{R}^{m\times n}$ be defined similarly. If

$$\mu\colon\mathbb{R}^{m\times p}\times\mathbb{R}^{p\times n}\to\mathbb{R}^{m\times n}$$

is matrix multiplication, then

$$\mu(e_{ij},\bar{e}_{k\ell})=e_{ij}\bar{e}_{k\ell}=\delta_{jk}\tilde{e}_{i\ell}.$$

For $\alpha = \sum \alpha^{ij} e_{ij} \in \Omega^*(M, \mathbb{R}^{m\times p})$ and $\beta = \sum \beta^{k\ell} \bar{e}_{k\ell} \in \Omega^*(M, \mathbb{R}^{p\times n})$, by Proposition 21.1,

$$\begin{aligned}
\alpha \cdot \beta &= \sum \alpha^{ij} \wedge \beta^{k\ell} \mu(e_{ij}, \bar{e}_{k\ell}) \\
&= \sum \alpha^{ij} \wedge \beta^{k\ell} \delta_{jk} \tilde{e}_{i\ell} \\
&= \sum_{i,\ell} \Big(\sum_k \alpha^{ik} \wedge \beta^{k\ell} \Big) \tilde{e}_{i\ell} = \alpha \wedge \beta,
\end{aligned}$$

the wedge product of matrices of forms defined in Section 11.1.

21.3 Directional Derivative of a Vector-Valued Function

If $f\colon M \to \mathbb{R}$ is a smooth function on the manifold M and X_p is a tangent vector to M at p, then we may interpret $X_p f$ as the directional derivative of f at p in the direction X_p; indeed, for any smooth curve $c(t)$ in M with initial point $c(0) = p$ and initial vector $c'(0) = X_p$,

$$X_p f = \left.\frac{d}{dt}\right|_{t=0} f(c(t)). \tag{21.5}$$

To extend the definition of $X_p f$ to a smooth vector-valued function $f\colon M \to V$, choose a basis $v_1, \dots, v_n$ for V and write $f = \sum f^i v_i$ for some smooth real-valued functions $f^i\colon M \to \mathbb{R}$. Define

$$X_p f = \sum (X_p f^i) v_i. \tag{21.6}$$

It is a routine exercise to show that the definition 21.6 is independent of the choice of basis (Problem 21.2). It follows that for a smooth vector-valued function $f\colon M \to V$ and the same smooth curve $c(t)$ in M as in (21.5),

$$\begin{aligned}
X_p f &= \sum (X_p f^i) v_i && \text{(by (21.6))} \\
&= \sum_i \left(\left.\frac{d}{dt}\right|_{t=0} f^i(c(t)) \right) v_i && \text{(by (21.5))} \\
&= \left.\frac{d}{dt}\right|_{t=0} \sum_i f^i(c(t)) v_i && \text{(by (21.6))} \\
&= \left.\frac{d}{dt}\right|_{t=0} f(c(t)).
\end{aligned}$$

This shows that for a vector-valued function f, one can still interpret $X_p f$ as the directional derivative of f at p in the direction X_p.

21.4 Exterior Derivative of a Vector-Valued Form

As before, M is a manifold and V is a finite-dimensional vector space. To define the exterior derivative of a V-valued k-form $\alpha \in \Omega^k(M, V)$, choose a basis $v_1, \dots, v_n$ for V, write $\alpha = \sum \alpha^i v_i$ with $\alpha^i \in \Omega^k(M)$, and define

$$d\alpha = \sum (d\alpha^i) v_i. \tag{21.7}$$

It is easy to check that so defined, the exterior derivative $d\alpha$ is independent of the choice of basis (Problem 21.3). This definition is consistent with the definition of the exterior derivative of a matrix-valued form in Section 11.1.

Proposition 21.3. *The exterior derivative on vector-valued forms on a manifold M is an antiderivation: if $\alpha \in \Omega^k(M,V)$, $\beta \in \Omega^\ell(M,W)$, and $\mu\colon V \times W \to Z$ is a bilinear map of vector spaces, then*

$$d(\alpha\cdot\beta) = (d\alpha)\cdot\beta + (-1)^{\deg\alpha}\alpha\cdot d\beta.$$

Proof. The easiest way to prove this is to write α and β in terms of bases for V and W, respectively, and to reduce the proposition to the antiderivation property of d on the wedge product of ordinary forms.

To carry this out, let $\{v_i\}$ be a basis for V and $\{w_j\}$ a basis for W. Then $\alpha = \sum \alpha^i v_i$ and $\beta = \sum \beta^j w_j$ for ordinary forms α^i, β^j on M. By Proposition 21.1,

$$\alpha\cdot\beta = \sum_{i,j} (\alpha^i \wedge \beta^j)\mu(v_i, w_j).$$

Then

$$\begin{aligned}
d(\alpha\cdot\beta) &= \sum_{i,j} d(\alpha^i\wedge\beta^j)\mu(v_i,w_j) \qquad \big(\text{by (21.7)}\big)\\
&= \sum_{i,j}\big((d\alpha^i)\wedge\beta^j\big)\,\mu(v_i,w_j) + (-1)^{\deg\alpha}\sum_{i,j}\big(\alpha^i\wedge d\beta^j\big)\,\mu(v_i,w_j)\\
&\qquad\qquad (\text{antiderivation property of } d \text{ on scalar forms})\\
&= \sum (d\alpha^i)v_i\cdot\sum\beta^j w_j + (-1)^{\deg\alpha}\sum\alpha^i v_i\cdot\sum(d\beta^j)w_j\\
&\qquad\qquad (\text{Proposition 21.1})\\
&= (d\alpha)\cdot\beta + (-1)^{\deg\alpha}\alpha\cdot d\beta.
\end{aligned}$$

□

If α and β are matrix-valued forms on a manifold, say of size $m\times p$ and $p\times n$, respectively, and

$$\mu\colon \mathbb{R}^{m\times p}\times\mathbb{R}^{p\times n}\to\mathbb{R}^{m\times n}$$

is matrix multiplication, then by Example 21.2, $\alpha\cdot\beta = \alpha\wedge\beta$, and Proposition 21.3 becomes the antiderivation formula for the matrix wedge product:

$$d(\alpha\wedge\beta) = (d\alpha)\wedge\beta + (-1)^{\deg\alpha}\alpha\wedge d\beta. \tag{21.8}$$

21.5 Differential Forms with Values in a Lie Algebra

In the cases of special interest to us, V is either a Lie algebra $\mathfrak{g}$ or more specifically the matrix algebra $\mathfrak{gl}(n,\mathbb{R})$. If $\mu\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is the Lie bracket, then we write $[\alpha,\beta]$ instead of $\alpha\cdot\beta$ and call it the ***Lie bracket*** of the two $\mathfrak{g}$-valued forms α and β. For $\alpha\in\Omega^k(M,\mathfrak{g})$, $\beta\in\Omega^\ell(M,\mathfrak{g})$, $p\in M$ and $u_1,\ldots,u_{k+\ell}\in T_pM$,

$$[\alpha,\beta]_p(u_1,\dots,u_{k+\ell}) = \sum_{\substack{(k,\ell)\text{-shuffles}\\ \sigma}} (\operatorname{sgn}\sigma)[\alpha_p(u_{\sigma(1)},\dots,u_{\sigma(k)}),\beta_p(u_{\sigma(k+1)},\dots,u_{\sigma(k+\ell)})]. \tag{21.9}$$

For example, if α and β are $\mathfrak{g}$-valued 1-forms on a manifold M, and X and Y are C^∞ vector fields on M, then by (21.9),

$$[\alpha,\beta](X,Y) = [\alpha(X),\beta(Y)] - [\alpha(Y),\beta(X)]. \tag{21.10}$$

In practice it is usually easier to calculate $[\alpha,\beta]$ using a basis. For $V=\mathfrak{g}$, Proposition 21.1 assumes the following form.

Proposition 21.4. *Suppose $\{A_1,\dots,A_n\}$ is a set of vectors in a Lie algebra $\mathfrak{g}$ and the forms $\alpha\in\Omega^k(M,\mathfrak{g})$ and $\beta\in\Omega^\ell(M,\mathfrak{g})$ can be written as $\alpha=\sum\alpha^iA_i$ and $\beta=\sum\beta^jA_j$. Then*

$$[\alpha,\beta] = \sum_{i,j}(\alpha^i\wedge\beta^j)[A_i,A_j] \in \Omega^{k+\ell}(M,\mathfrak{g}). \tag{21.11}$$

Note that in this proposition $\{A_1,\dots,A_n\}$ is an arbitrary set of vectors in $\mathfrak{g}$; it need not be a basis.

Proposition 21.5. *If α is a $\mathfrak{g}$-valued k-form and β a $\mathfrak{g}$-valued ℓ-form on a manifold, then*

$$[\alpha,\beta] = (-1)^{k\ell+1}[\beta,\alpha].$$

Proof. Let $B_1,\dots,B_n$ be a basis for $\mathfrak{g}$. Then $\alpha=\sum\alpha^iB_i$ and $\beta=\sum\beta^jB_j$ for some $\mathbb{R}$-valued forms α^i and β^j. Then

$$\begin{aligned}
[\alpha,\beta] &= \sum_{i,j}(\alpha^i\wedge\beta^j)[B_i,B_j] \\
&= (-1)^{k\ell+1}\sum_{i,j}(\beta^j\wedge\alpha^i)[B_j,B_i] \\
&\qquad \text{(since } \wedge \text{ is graded commutative and } [\ ,\ ] \text{ is skew)} \\
&= (-1)^{k\ell+1}[\beta,\alpha].
\end{aligned}$$ □

Therefore, if α and β are $\mathfrak{g}$-valued 1-forms, then $[\alpha,\beta]$ is symmetric in α and β and $[\alpha,\alpha]$ is not necessarily zero. For the Lie bracket of $\mathfrak{g}$-valued forms, Proposition 21.3 can be restated as follows.

Proposition 21.6. *The exterior derivative d on $\Omega^*(M,\mathfrak{g})$ is an antiderivation with respect to the Lie bracket: if α and β are homogeneous $\mathfrak{g}$-valued forms on a manifold M, then*

$$d[\alpha,\beta] = [d\alpha,\beta] + (-1)^{\deg\alpha}[\alpha,d\beta].$$

Differential Forms with Values in $\mathfrak{gl}(n,\mathbb{R})$

On the Lie algebra $\mathfrak{gl}(n,\mathbb{R})$ of $n \times n$ real matrices, there are two natural bilinear products, matrix multiplication and the Lie bracket. They are related by

$$[A,B] = AB - BA \qquad \text{for } A,B \in \mathfrak{gl}(n,\mathbb{R}).$$

Let e_{ij} be the n by n matrix with a 1 in the (i,j) entry and 0 everywhere else. Then

$$e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell},$$

where δ_{jk} is the Kronecker delta.

For two homogeneous $\mathfrak{gl}(n,\mathbb{R})$-valued forms α and β on a manifold M, we denote their products induced from matrix multiplication and from the Lie bracket by $\alpha \wedge \beta$ and $[\alpha,\beta]$, respectively. In terms of the basis $\{e_{ij}\}$ for $\mathfrak{gl}(n,\mathbb{R})$,

$$\alpha = \sum \alpha^{ij} e_{ij}, \qquad \beta = \sum \beta^{k\ell} e_{k\ell}$$

for some real-valued forms α^{ij} and $\beta^{k\ell}$ on M. By Proposition 21.1 the wedge product of α and β is

$$\alpha \wedge \beta = \sum_{i,j,k,\ell} \alpha^{ij} \wedge \beta^{k\ell} e_{ij}e_{k\ell} = \sum_{i,k,\ell} \alpha^{ik} \wedge \beta^{k\ell} e_{i\ell},$$

while their Lie bracket is

$$[\alpha,\beta] = \sum \alpha^{ij} \wedge \beta^{k\ell} [e_{ij}, e_{k\ell}].$$

Proposition 21.7. *If α and β are homogeneous $\mathfrak{gl}(n,\mathbb{R})$-valued forms on a manifold M, then*

$$[\alpha,\beta] = \alpha \wedge \beta - (-1)^{(\deg\alpha)(\deg\beta)} \beta \wedge \alpha.$$

Proof. Problem 21.4. □

Hence, if α is a homogeneous $\mathfrak{gl}(n,\mathbb{R})$-valued form on a manifold, then

$$[\alpha,\alpha] = \begin{cases} 2\,\alpha \wedge \alpha & \text{if } \deg\alpha \text{ is odd,} \\ 0 & \text{if } \deg\alpha \text{ is even.} \end{cases} \qquad (21.12)$$

21.6 Pullback of Vector-Valued Forms

Like scalar-valued forms, vector-valued forms on a manifold can be pulled back by smooth maps. Suppose V is a vector space, α a C^∞ V-valued k-form on a manifold M, and $f\colon N \to M$ a C^∞ map. The ***pullback*** $f^*\alpha$ is the V-valued k-form on N defined as follows: for any $p \in N$ and $u_1, \dots, u_k \in T_pN$,

$$(f^*\alpha)_p(u_1,\dots,u_k) = \alpha_{f(p)}(f_*u_1,\dots,f_*u_k).$$

The pullback of vector-valued differential forms offers no surprises; it satisfies the same properties as the pullback of scalar-valued differential forms.

Proposition 21.8. *Suppose $\mu\colon V\times W\to Z$ is a bilinear map of finite-dimensional vector spaces, and $f\colon N\to M$ is a smooth map of manifolds.*

(i) *If $\{v_1,\dots,v_n\}$ is a set in the vector space V and $\alpha=\sum\alpha^i v_i$ for $\alpha^i\in\Omega^k(M)$, then $f^*\alpha=\sum(f^*\alpha^i)v_i$.*
(ii) *The pullback f^* of vector-valued forms commutes with the product: if $\alpha\in\Omega^k(M,V)$ and $\beta\in\Omega^\ell(M,W)$, then*

$$f^*(\alpha\cdot\beta)=(f^*\alpha)\cdot(f^*\beta).$$

(iii) *The pullback f^* commutes with the exterior derivative: for $\alpha\in\Omega^k(M,V)$, $f^*d\alpha=df^*\alpha$.*

Proof. All three properties are straightforward to prove. We leave the proof as an exercise (Problem 21.6). □

In particular, if $\mu\colon\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is the Lie bracket of a Lie algebra $\mathfrak{g}$, then

$$f^*[\alpha,\beta]=[f^*\alpha,f^*\beta]$$

for any $\mathfrak{g}$-valued forms α,β on M.

21.7 Forms with Values in a Vector Bundle

For a vector space V, a V-valued k-form on a manifold M assigns to each point $p\in M$ an alternating k-linear map

$$T_pM\times\cdots\times T_pM\to V.$$

More generally, we may allow the vector space V to vary from point to point. If E is a vector bundle over M, then an E-valued k-form assigns to each point p in M an alternating k-linear map

$$T_pM\times\cdots\times T_pM\to E_p.$$

Reasoning as above, we conclude that an E-valued k-form is a section of the vector bundle $\left(\bigwedge^k T^*M\right)\otimes E$. The space of smooth E-valued k-forms on M is denoted by

$$\Omega^k(M,E):=\Gamma\left(\left(\bigwedge^k T^*M\right)\otimes E\right).$$

Example 21.9 (*The curvature of a connection*). Let $E\to M$ be a vector bundle with a connection ∇. In Section 10.3 we saw that the curvature of the connection assigns to each point $p\in M$ an alternating bilinear map

$$T_pM\times T_pM\to\operatorname{End}(E_p).$$

Thus, the curvature is a section of the vector bundle $(\bigwedge^2 T^*M)\otimes\operatorname{End}(E)$. It is in fact a C^∞ section (Problem 21.7). As such, it is a smooth 2-form on M with values in the vector bundle $\operatorname{End}(E)$.

21.8 Tensor Fields on a Manifold

Vector fields and differential forms are examples of tensor fields on a manifold.

Definition 21.10. An (a,b)-***tensor field*** on a manifold M is a section of the vector bundle

$$\left(\bigotimes^a TM\right) \otimes \left(\bigotimes^b T^*M\right) = \underbrace{TM \otimes \cdots \otimes TM}_{a} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{b},$$

where the tangent bundle TM occurs a times and the cotangent bundle T^*M occurs b times.

Example. A Riemannian metric on M is a section of $T^*M \otimes T^*M$, so it is a $(0,2)$-tensor field.

Example. As a section of the tangent bundle TM, a vector field on M is a $(1,0)$-tensor field.

Example. A differential k-form on M associates to each point $p \in M$ an alternating k-linear map

$$T_pM \times \cdots \times T_pM \to \mathbb{R}.$$

By the universal mapping theorem, a k-linear map on T_pM corresponds to a linear map

$$T_pM \otimes \cdots \otimes T_pM \to \mathbb{R}.$$

Hence, a k-form is a section of $\bigotimes^k T^*M$ corresponding to an alternating map. In particular, it is a $(0,k)$-tensor field. (A k-form is also a section of $\bigwedge^k T^*M \to M$.)

More generally, if E is a vector bundle over M, then an E-***valued*** (a,b)-***tensor field*** is a section of

$$\left(\bigotimes^a TM\right) \otimes \left(\bigotimes^b T^*M\right) \otimes E.$$

Example. By Example 21.9, the curvature of a connection on a vector bundle $E \to M$ is a section of the vector bundle

$$\begin{aligned}\operatorname{Hom}(TM \otimes TM, \operatorname{End}(E)) &\simeq (TM \otimes TM)^\vee \otimes \operatorname{End}(E) \quad &\text{(by Prop. 18.14)}\\ &\simeq T^*M \otimes T^*M \otimes \operatorname{End}(E) \quad &\text{(by Prop. 18.15).}\end{aligned}$$

As such, it is an $\operatorname{End}(E)$-valued $(0,2)$-tensor field on the manifold M.

21.9 The Tensor Criterion

A tensor field $T \in \Gamma\big(T^{a,b}(M)\big)$ on a manifold M defines an $\mathcal{F}$-multilinear function

$$\Omega^1(M)^a \times \mathfrak{X}(M)^b \to \mathcal{F},$$
$$(\omega_1,\dots,\omega_a,Y_1,\dots,Y_b) \mapsto T(\omega_1,\dots,\omega_a,Y_1,\dots,Y_b).$$

Conversely, the following proposition gives a condition for a multilinear function to be a tensor field.

Proposition 21.11 (The tensor criterion). *There is a one-to-one correspondence between*

$$\{\mathcal{F}\text{-}\textit{multilinear functions } T : \Omega^1(M)^a \times \mathfrak{X}(M)^b \to \mathcal{F}\}$$

and

$$\{\ \textit{tensor fields } \tilde{T} \in \Gamma\big(T^{a,b}(M)\big)\}$$

such that

$$T(\omega_1,\dots,\omega_a,Y_1,\dots,Y_b)(p) = \tilde{T}_p(\omega_{1,p},\dots,\omega_{a,p},Y_{1,p},\dots,Y_{b,p}).$$

Because of this proposition, we often identify an $\mathcal{F}$-multilinear function T with the tensor field $\tilde{T}$ that it corresponds to.

Proof. We give the proof only for $(a,b) = (2,1)$, since the general case is similar. Suppose

$$T\colon \Omega^1(M) \times \Omega^1(M) \times \mathfrak{X}(M) \to \mathcal{F}$$

is 3-linear over $\mathcal{F}$. Since $\Omega^1(M) = \Gamma(T^*M)$ and $\mathfrak{X}(M) = \Gamma(TM)$, by Proposition 7.28, for each $p \in M$ there is a unique $\mathbb{R}$-multilinear map

$$T_p\colon T_p^*M \times T_p^*M \times T_pM \to \mathbb{R}$$

such that for all $(\omega_1,\omega_2,Y) \in \Gamma(T^*M) \times \Gamma(T^*M) \times \Gamma(TM)$,

$$T_p(\omega_{1,p},\omega_{2,p},Y_p) = T(\omega_1,\omega_2,Y)(p).$$

By the universal property of the tensor product, the multilinear map T_p corresponds to a unique linear map $\tilde{T}_p\colon T_p^*M \otimes T_p^*M \otimes T_pM \to \mathbb{R}$ such that

$$\tilde{T}_p(\omega_{1,p} \otimes \omega_{2,p} \otimes Y_p) = T(\omega_1,\omega_2,Y)(p).$$

In other words, $\tilde{T}_p$ is an element of

$$\begin{aligned}\operatorname{Hom}(T_p^*M \otimes T_p^*M \otimes T_pM, \mathbb{R}) &\simeq (T_p^*M \otimes T_p^*M \otimes T_pM)^\vee \\ &= T_pM \otimes T_pM \otimes T_p^*M \quad \text{(by Prop. 18.15).}\end{aligned}$$

Thus, $\tilde{T}$ is a section of $T^{2,1}(M) = TM \otimes TM \otimes T^*M$. □

21.10 Remark on Signs Concerning Vector-Valued Forms

Formulas involving vector-valued forms often have additional signs that their scalar-valued or constant analogues do not have. For example, the Lie bracket is anticommutative on elements of a Lie algebra $\mathfrak{g}$:

$$[A,B] = -[B,A],$$

but when applied to $\mathfrak{g}$-valued forms,

$$[\alpha,\beta] = (-1)^{(\deg\alpha)(\deg\beta)+1}[\beta,\alpha]. \tag{21.13}$$

There is a general rule that appears to describe the additional sign or at least to serve as a useful mnemonic: whenever two graded objects x and y are interchanged, the additional sign is $(-1)^{\deg x \deg y}$. This applies to example (21.13) above. It also applies to Proposition 21.7: For $A,B \in \mathfrak{gl}(n,\mathbb{R})$,

$$[A,B] = AB - BA,$$

but for $\mathfrak{gl}(n,\mathbb{R})$-valued forms,

$$[\alpha,\beta] = \alpha\wedge\beta - (-1)^{\deg\alpha\deg\beta}\beta\wedge\alpha.$$

If we assign a degree of 1 to the exterior derivative d, then the antiderivation property in Proposition 21.3 also fits this pattern.

Problems

21.1. Associativity of the product of vector-valued forms
Suppose $\mu\colon V\times W \to Z$ is a bilinear map of vector spaces, $\lambda \in \Omega^{\ell}(M)$, $\alpha \in \Omega^{a}(M,V)$, and $\beta \in \Omega^{b}(M,W)$. Prove that

$$(\lambda\cdot\alpha)\cdot\beta = \lambda\cdot(\alpha\cdot\beta) = (-1)^{\ell\cdot a}\alpha\cdot(\lambda\cdot\beta).$$

(*Hint*: Write α and β in terms of a basis for V and a basis for W, respectively, and apply Proposition 21.1.)

21.2. Directional derivative of a vector-valued function
Show that the definition (21.6) of the directional derivative is independent of the choice of basis.

21.3. Exterior derivative of a vector-valued function
Prove that the definition (21.7) of the exterior derivative is independent of the choice of basis.

21.4. Lie bracket in terms of the wedge product
Prove Proposition 21.7.

21.5. Triple product of 1-forms
Show that if α is any $\mathfrak{g}$-valued 1-form on a manifold, then

$$[[\alpha,\alpha],\alpha] = 0.$$

21.6. Pullback of vector-valued forms
Prove Proposition 21.8.

21.7. Curvature as a section
Let ∇ be a connection on a smooth vector bundle $\pi\colon E \to M$. Show that the curvature of ∇ is a ***smooth*** section of the vector bundle $\left(\bigwedge^2 T^*M\right) \otimes \operatorname{End}(E)$.

21.8. Exterior derivative of a vector-valued 1-form
Let V be a vector space. Prove that for a C^∞ V-valued 1-form ω and C^∞ vector fields X,Y on M,

$$(d\omega)(X,Y) = X\omega(Y) - Y\omega(X) - \omega\big([X,Y]\big).$$

(*Hint*: See [21, Th. 20.14, p. 233].)

21.9.* Maurer–Cartan equation
The ***Maurer–Cartan form*** on a Lie group G with Lie algebra $\mathfrak{g}$ is the unique $\mathfrak{g}$-valued 1-form θ on G that is the identity map at the identity element $e \in G$. Thus, for $X_e \in T_eG$ and $X_g \in T_gG$,

$$\theta_e(X_e) = X_e \in \mathfrak{g}$$

and

$$\theta_g(X_g) = (\ell_{g^{-1}}^*\theta_e)(X_g) = \theta_e(\ell_{g^{-1}*}X_g) = \ell_{g^{-1}*}X_g.$$

Prove that the Maurer–Cartan form satisfies the Maurer–Cartan equation

$$d\theta + \frac{1}{2}[\theta,\theta] = 0.$$

(*Hint*: By $\mathcal{F}$-linearity, it suffices to check this equation on left-invariant vector fields.)

21.10. Right translation of the Maurer–Cartan form
Prove that under right translation the Maurer–Cartan form θ on a Lie group G satisfies

$$r_g^*\theta = (\operatorname{Ad} g^{-1})\theta \quad \text{for } g \in G.$$

Chapter 5

Vector Bundles and Characteristic Classes

The Gauss–Bonnet theorem for a compact oriented Riemannian 2-manifold M may be stated in the following way:

$$\int_M \frac{1}{2\pi} K \operatorname{vol} = \chi(M),$$

where K is the Gaussian curvature. What is especially significant about this theorem is that on the left-hand side the 2-form $K \operatorname{vol}$ is locally the curvature form Ω_2^1 relative to an orthonormal frame of the Riemannian metric, but on the right-hand side the Euler characteristic $\chi(M)$ is a diffeomorphism invariant, independent of the Riemannian structure. Thus, the Gauss–Bonnet theorem for surfaces raises two interesting questions about higher-dimensional compact oriented manifolds:

(i) Is there a differential form whose integral over M gives the Euler characteristic?
(ii) Is it possible to construct from the curvature tensor diffeomorphism invariants other than the Euler characteristic?

Let $H^i(M)$ be the de Rham cohomology vector space of M in degree i. The ith ***Betti number*** b_i of M is the dimension of $H^i(M)$. In algebraic topology one learns that the Euler characteristic of a compact manifold may be computed as the alternating sum of the Betti numbers [18, Th. 22.2, p. 124]:

$$\chi(M) = \sum_{i=0}^{\dim M} (-1)^i b_i. \tag{*}$$

If the manifold is not compact, this may be an infinite sum and the Euler characteristic need not be defined.

Furthermore, the Poincaré duality theorem for a compact orientable manifold asserts that the integral of the wedge product of forms of complementary dimensions

$$H^i(M) \times H^{n-i}(M) \xrightarrow{\wedge} H^n(M) \xrightarrow{\int} \mathbb{R},$$
$$([\omega],[\tau]) \mapsto \int_M \omega \wedge \tau,$$

L.W. Tu, *Differential Geometry*, Graduate Texts in Mathematics 275,
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is a nondegenerate pairing [3, p. 44]. Hence, by Lemma 19.11, there is a linear isomorphism of $H^i(M)$ with the dual of $H^{n-i}(M)$:

$$H^i(M) \simeq H^{n-i}(M)^\vee$$

and

$$b^i = b^{n-i}.$$

Example. If M is a compact orientable manifold of dimension 5, then its Betti numbers pair up:

$$b_0 = b_5, \quad b_1 = b_4, \quad b_2 = b_3$$

and so

$$\begin{aligned} \chi(M) &= b_0 - b_1 + b_2 - b_3 + b_4 - b_5 \\ &= (b_0 - b_5) + (b_2 - b_3) + (b_4 - b_1) \\ &= 0. \end{aligned}$$

This example generalizes to any odd-dimensional compact orientable manifold.

Proposition. *The Euler characteristic of a compact orientable odd-dimensional manifold is* 0.

Proof. If $n = \dim M$ is odd, then the Betti numbers b_i and b_{n-i} are equal by Poincaré duality. Moreover, they occur with opposite signs in the alternating sum for the Euler characteristic. Hence, all the terms in (*) cancel out and $\chi(M) = 0$. □

Thus, from the point of view of generalizing the Gauss–Bonnet theorem by answering question (i), the only manifolds of interest are compact oriented Riemannian manifolds of *even* dimension.

Following Chern and Weil, we will associate to a vector bundle E over M global differential forms on M constructed from the curvature matrix of a connection E. These forms turn out to be closed; moreover, their cohomology classes are independent of the connection and so are diffeomorphism invariants of the vector bundle E. They are called ***characteristic classes*** of E. Specializing to the tangent bundle TM, we obtain in this way new diffeomorphism invariants of the manifold M, among which are the ***Pontrjagin classes*** and ***Pontrjagin numbers*** of M.

§22 Connections and Curvature Again

This section collects together some facts about connections and curvature on a vector bundle that will be needed in the construction of characteristic classes. We first study how connection and curvature matrices transform under a change of frame. Then we derive the Bianchi identities, which are formulas for the exterior derivative of the connection and curvature matrices. Just as there are two structural equations, corresponding to the torsion vector and the curvature matrix, so there are two Bianchi identities, corresponding to the exterior derivatives of the two structural equations.

22.1 Connection and Curvature Matrices Under a Change of Frame

In Section 10.6 we showed that a connection on a vector bundle $E \to M$ may be restricted to a connection over any open subset U of M:

$$\nabla^U : \mathfrak{X}(U) \times \Gamma(U,E) \to \Gamma(U,E).$$

We will usually omit the superscript U and write ∇ for ∇^U.

Suppose there is a frame $e_1,\dots,e_r$ for E over U. Then the connection matrix ω of ∇ relative to the frame $e_1,\dots,e_r$ over U is the matrix $[\omega^i_j]$ of 1-forms defined by

$$\nabla_X e_j = \sum \omega^i_j(X) e_i,$$

for $X \in \mathfrak{X}(U)$. If we write the frame $e_1,\dots,e_r$ as a row vector $e = [e_1 \ \cdots \ e_r]$, then in matrix notation

$$\nabla_X e = e\omega(X).$$

As a function of X,

$$\nabla e = e\omega.$$

Similarly, for C^∞ vector fields X,Y on U, the curvature matrix Ω of the connection ∇ is the matrix $[\Omega^i_j]$ of 2-forms defined by

$$R(X,Y)e_j = \sum \Omega^i_j(X,Y)e_i.$$

We next study how the connection and curvature matrices ω and Ω transform under a change of frame. Suppose $\bar{e}_1,\dots,\bar{e}_r$ is another frame for E over U. Let $\bar{\omega} = [\bar{\omega}^i_j]$ and $\bar{\Omega} = [\bar{\Omega}^i_j]$ be the connection and curvature matrices of the connection ∇ relative to this new frame. At each point p, the basis vector $\bar{e}_\ell(p)$ is a linear combination of $e_1(p),\dots,e_r(p)$:

$$\bar{e}_\ell(p) = \sum a^k_\ell(p) e_k(p).$$

As sections on U,

$$\bar{e}_\ell = \sum a^k_\ell e_k.$$

So we get a matrix of functions $a = [a^k_\ell]$. As the coefficients of a smooth section with respect to a smooth frame, the a^k_ℓ are smooth functions on U. At each point p, the matrix $a(p) = [a^k_\ell(p)]$ is invertible because it is a change of basis matrix. Thus we can think of a as a C^∞ function $U \to \mathrm{GL}(r,\mathbb{R})$. As row vectors,

$$[\bar{e}_1 \ \cdots \ \bar{e}_r] = [e_1 \ \cdots \ e_r][a^k_\ell],$$

or in matrix notation,

$$\bar{e} = ea. \tag{22.1}$$

Since in matrix notation (22.1), the matrix of functions a^i_j must be written on the right of the row vector of sections e_i, we will recast the Leibniz rule in this form.

Suppose s is a C^∞ section of the vector bundle E and f is a C^∞ function on M. By the Leibniz rule, for any C^∞ vector field $X \in \mathfrak{X}(M)$,

$$\begin{aligned}\nabla_X(sf) &= (\nabla_X s)f + sX(f)\\ &= (\nabla_X s)f + s\,df(X).\end{aligned} \tag{22.2}$$

We may view ∇s as a function $\mathfrak{X}(M) \to \Gamma(E)$ with $(\nabla s)(X) = \nabla_X s$. Then we may suppress the argument X from (22.2) and rewrite the Leibniz rule as

$$\nabla(sf) = (\nabla s)f + s\,df \quad \text{for } s \in \Gamma(E),\ f \in C^\infty(M). \tag{22.3}$$

Theorem 22.1. *Suppose e and $\bar{e}$ are two frames for the vector bundle E over U such that $\bar{e} = ea$ for some $a\colon U \to \mathrm{GL}(r,\mathbb{R})$. If ω and $\bar{\omega}$ are the connection matrices and Ω and $\bar{\Omega}$ are the curvature matrices of a connection ∇ relative to the two frames, then*

(i) $\bar{\omega} = a^{-1}\omega a + a^{-1}da$,
(ii) $\bar{\Omega} = a^{-1}\Omega a$.

Proof. (i) We use the Leibniz rule (22.3) to derive the transformation rule for the connection matrix under a change of frame. Since $\bar{e} = ea$, we have $e = \bar{e}a^{-1}$. Recall that if $a = [a^i_j]$ is a matrix of C^∞ functions, then da is the matrix of C^∞ 1-forms $[da^i_j]$. Thus,

$$\begin{aligned}\nabla\bar{e} &= \nabla(ea)\\ &= (\nabla e)a + e\,da && \text{(Leibniz rule)}\\ &= (e\omega)a + e\,da && (\nabla e = e\omega)\\ &= \bar{e}a^{-1}\omega a + \bar{e}a^{-1}da && (e = \bar{e}a^{-1})\\ &= \bar{e}(a^{-1}\omega a + a^{-1}da).\end{aligned}$$

Therefore,

$$\bar{\omega} = a^{-1}\omega a + a^{-1}da.$$

(ii) Since the structural equation (Theorem 11.1)

$$\bar{\Omega} = d\bar{\omega} + \bar{\omega}\wedge\bar{\omega} \tag{22.4}$$

expresses the curvature $\bar{\Omega}$ in terms of the connection matrix ω, the brute-force way to obtain the transformation rule for $\bar{\Omega}$ is to plug the formula in part (i) into (22.4), expand the terms, and try to write the answer in terms of Ω.

Another way is to note that since for any $X_p, Y_p \in T_pM$, the curvature $R(X_p,Y_p)$ is the linear transformation of E_p to E_p with matrix $[\Omega^i_j(X_p,Y_p)]$ relative to the basis $e_1,\dots,e_r$ at p, a change of basis should lead to a conjugate matrix. Indeed, for $X, Y \in \mathfrak{X}(U)$,

$$R(X,Y)e_j = \sum_i \Omega^i_j(X,Y)e_i,$$

or in matrix notation,

$$R(X,Y)e = R(X,Y)[e_1,\dots,e_r] = [e_1,\dots,e_r][\Omega^i_j(X,Y)] = e\Omega(X,Y).$$

Suppressing X,Y, we get

$$R(e) = e\Omega.$$

Hence,

$$\begin{aligned} R(\bar{e}) &= R(ea) \\ &= R(e)a \qquad (R \text{ is } \mathcal{F}\text{-linear in the argument } e) \\ &= e\Omega a \\ &= \bar{e}a^{-1}\Omega a. \end{aligned}$$

This proves that $\bar{\Omega} = a^{-1}\Omega a$. □

22.2 Bianchi Identities

When $E \to M$ is the tangent bundle, a connection on E is an affine connection as defined in Section 6.1. For an affine connection ∇, the torsion

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

is defined and is a tensor. If $e_1,\dots,e_n$ is a frame for the tangent bundle TM over U, then the torsion forms τ^i are given by

$$T(X,Y) = \sum \tau^i(X,Y)e_i.$$

We collect $\tau^1,\dots,\tau^n$ into a column vector $\tau = [\tau^i]$ of 2-forms. Similarly, if $\theta^1,\dots,\theta^n$ are the 1-forms dual to the basis $e_1,\dots,e_n$, then θ is the column vector $[\theta^i]$. Recall the two structural equations (Theorem 11.7):



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(1856–1928)

$$\begin{aligned} &(1) \quad \tau = d\theta + \omega\wedge\theta, \\ &(2) \quad \Omega = d\omega + \omega\wedge\omega. \end{aligned}$$

Proposition 22.2 (First Bianchi identity). *Let ∇ be a connection on the tangent bundle TM of a manifold M. Suppose θ and τ are the column vectors of dual 1-forms and torsion forms, respectively, on a framed open set, and ω and Ω are the connection and curvature matrices respectively. Then*

$$d\tau = \Omega\wedge\theta - \omega\wedge\tau.$$

Proof. Differentiating the first structural equation gives

$$\begin{aligned} d\tau &= d(d\theta) + d\omega\wedge\theta - \omega\wedge d\theta \\ &= 0 + (\Omega - \omega\wedge\omega)\wedge\theta - \omega\wedge(\tau - \omega\wedge\theta) \\ &\quad \text{(rewriting } d\omega \text{ and } d\theta \text{ using the two structural equations)} \\ &= \Omega\wedge\theta - \omega\wedge\tau \quad \text{(since } \omega\wedge\omega\wedge\theta \text{ cancels out).} \end{aligned}$$ □

Like the second structural equation, the next two identities apply more generally to a connection ∇ on any smooth vector bundle, not just the tangent bundle.

Proposition 22.3 (Second Bianchi identity). *Let ∇ be a connection on a smooth vector bundle E. Suppose ω and Ω are the connection and curvature matrices of ∇ relative to a frame for E over an open set. Then*

$$d\Omega = \Omega\wedge\omega - \omega\wedge\Omega.$$

Proof. Differentiating the second structural equation gives

$$\begin{aligned} d\Omega &= d(d\omega) + (d\omega)\wedge\omega - \omega\wedge d\omega \\ &= 0 + (\Omega - \omega\wedge\omega)\wedge\omega - \omega\wedge(\Omega - \omega\wedge\omega) \\ &\quad \text{(rewriting } d\omega \text{ using the second structural equation)} \\ &= \Omega\wedge\omega - \omega\wedge\Omega. \end{aligned}$$ □

We will use the notation Ω^k to mean the wedge product $\Omega\wedge\cdots\wedge\Omega$ of the curvature matrix k times.

Proposition 22.4 (Generalized second Bianchi identity). *Under the same hypotheses as in Proposition 22.3, for any integer $k \geq 1$,*

$$d(\Omega^k) = \Omega^k\wedge\omega - \omega\wedge\Omega^k.$$

Proof. Problem 22.3. □

22.3 The First Bianchi Identity in Vector Form

For a Riemannian connection, the torsion form τ is zero and the first Bianchi identity (Proposition 22.2) simplifies to

$$\Omega\wedge\theta = 0.$$

This identity translates into a symmetry property of the curvature tensor.

Theorem 22.5. *If X, Y, Z are C^∞ vector fields on a Riemannian manifold M, then the curvature tensor of the Riemannian connection satisfies*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

Proof. Let $e_1, \ldots, e_n$ be a frame over an open set U in M, and $\theta^1, \ldots, \theta^n$ the dual 1-forms. On U, $Z = \sum_j \theta^j(Z) e_j$. In the notation of the preceding section,

$$\begin{aligned} R(X,Y)Z &= \sum_j R(X,Y)\theta^j(Z)e_j \\ &= \sum_{i,j} \Omega^i_j(X,Y)\theta^j(Z)e_i. \end{aligned}$$

The ith component of $\Omega \wedge \theta$ is $\sum_j \Omega^i_j \wedge \theta^j$. Thus, for each i, the first Bianchi identity gives

$$\begin{aligned} 0 &= \sum_j (\Omega^i_j \wedge \theta^j)(X,Y,Z) \\ &= \sum_j \Omega^i_j(X,Y)\theta^j(Z) - \Omega^i_j(X,Z)\theta^j(Y) + \Omega^i_j(Y,Z)\theta^j(X). \end{aligned}$$

Multiplying by e_i and summing over i, we obtain

$$0 = R(X,Y)Z + R(Z,X)Y + R(Y,Z)X. \qquad \square$$

22.4 Symmetry Properties of the Curvature Tensor

With the first Bianchi identity, we have now proven three symmetry properties of the curvature tensor: Let X,Y,Z,W be C^∞ vector fields on a Riemannian manifold. Then the curvature tensor of the Riemannian connection satisfies

(i) (skew-symmetry in X and Y) $R(X,Y) = -R(Y,X)$.
(ii) (skew-symmetry in Z and W, Proposition 12.5)

$$\langle R(X,Y)Z,W\rangle = -\langle R(X,Y)W,Z\rangle.$$

(iii) (first Bianchi identity)

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

As an algebraic consequence of these three properties, a fourth symmetry property follows.

Theorem 22.6. *If X,Y,Z,W are C^∞ vector fields on a Riemannian manifold, then the curvature tensor of the Riemannian connection satisfies*

$$\langle R(X,Y)Z,W\rangle = \langle R(Z,W)X,Y\rangle.$$

Proof. By the first Bianchi identity,

$$\langle R(X,Y)Z,W\rangle + \langle R(Y,Z)X,W\rangle + \langle R(Z,X)Y,W\rangle = 0. \tag{22.5}$$

To cancel out the term $\langle R(Y,Z)X,W\rangle$ in (22.5), we add to (22.5) the first Bianchi identity starting with the term $\langle R(Y,Z)W,X\rangle$:

$$\langle R(Y,Z)W,X\rangle + \langle R(X,W)Y,X\rangle + \langle R(W,Y)Z,X\rangle = 0. \tag{22.6}$$

Similarly, to cancel out the term $\langle R(Z,X)Y,W\rangle$ in (22.5), we add

$$\langle R(Z,X)W,Y\rangle + \langle R(X,W)Z,Y\rangle + \langle R(W,Z)X,Y\rangle = 0. \tag{22.7}$$

Finally, to cancel out $\langle R(W,Y)Z,X\rangle$ in (22.6), we add

$$\langle R(W,Y)X,Z\rangle + \langle R(Y,X)W,Z\rangle + \langle R(X,W)Y,Z\rangle = 0. \tag{22.8}$$

Adding up the four equations (22.5), (22.6), (22.7), (22.8) and making use of the skew-symmetry properties, we get

$$2\langle R(X,Y)Z,W\rangle - 2\langle R(Z,W)X,Y\rangle = 0,$$

which proves the theorem. □

22.5 Covariant Derivative of Tensor Fields

A connection ∇ on the tangent bundle TM of a manifold M induces a covariant derivative on all tensor fields. This is the content of the following proposition.

Proposition 22.7. *Let ∇ be a connection and X, Y C^∞ vector fields on the manifold M.*

(i) *If ω is a C^∞ 1-form on M, then $\nabla_X\omega$ defined by*

$$(\nabla_X\omega)(Y) := X\big(\omega(Y)\big) - \omega(\nabla_X Y)$$

is a C^∞ 1-form on M.

(ii) *If T is a C^∞ (a,b)-tensor field on M, then $\nabla_X T$ defined by*

$$\begin{aligned}(\nabla_X T)(\omega_1,\dots,\omega_a,Y_1,\dots,Y_b) :=& X\big(T(\omega_1,\dots,\omega_a,Y_1,\dots,Y_b)\big)\\ &- \sum_{i=1}^{a} T(\omega_1,\dots,\nabla_X\omega_i,\dots\omega_a,Y_1,\dots,Y_b)\\ &- \sum_{j=1}^{b} T(\omega_1,\dots,\omega_a,Y_1,\dots,\nabla_X Y_j,\dots Y_b),\end{aligned}$$

for $\omega_i \in \Omega^1(M)$, $Y_j \in \mathfrak{X}(M)$, is a C^∞ (a,b)-tensor field on M.

Proof. By the tensor criterion (Proposition 21.11), it suffices to check that $\nabla_X\omega$ and $\nabla_X T$ are $\mathcal{F}$-linear in its arguments, where $\mathcal{F} := C^\infty(M)$ is the ring of C^∞ functions on M. Let $f \in \mathcal{F}$.

(i) By definition and the Leibniz rule in Y of a connection,

$$\begin{aligned}(\nabla_X\omega)(fY) &= X\big(f\omega(Y)\big) - \omega(\nabla_X fY)\\ &= (Xf)\omega(Y) + fX\big(\omega(Y)\big) - \omega\big((Xf)Y\big) - f\omega(\nabla_X Y)\\ &= f(\nabla_X\omega)(Y).\end{aligned}$$

(ii) Replace ω_i by $f\omega_i$ in the definition of $\nabla_X T$. On the right-hand side, we have

$$X\big(T(\omega_1,\dots,f\omega_i,\dots\omega_a,Y_1,\dots,Y_b)\big)$$
$$= (Xf)T(\omega_1,\dots,\omega_a,Y_1,\dots,Y_b) + fX\big(T(\omega_1,\dots,\omega_a,Y_1,\dots,Y_b)\big) \quad (22.9)$$

and

$$T(\omega_1,\dots,\nabla_X f\omega_i,\dots,\omega_a,Y_1,\dots,Y_b)$$
$$= T(\omega_1,\dots,(Xf)\omega_i,\dots,\omega_a,Y_1,\dots,Y_b) + fT(\omega_1,\dots,\nabla_X\omega_i,\dots\omega_a,Y_1,\dots,Y_b). \quad (22.10)$$

The first terms of these two expressions cancel out. All the other terms on the right of (22.9) and (22.10) are $\mathcal{F}$-linear in the ith argument.
Similarly, the right-hand side of the definition of $\nabla_X T$ is $\mathcal{F}$-linear in the argument Y_j.

□

On C^∞ functions f, we define $\nabla_X f = Xf$.
Let $T^{a,b}(M) = \left(\bigotimes^a TM\right) \otimes \left(\bigotimes^b T^*M\right)$ be the bundle of (a,b)-tensors on M.

Theorem 22.8. *Let ∇ be a connection on a manifold and $X \in \mathfrak{X}(M)$. The covariant derivative $\nabla_X : \Gamma\big(\oplus_{a,b=0}^\infty T^{a,b}(M)\big) \to \Gamma\big(\oplus_{a,b=0}^\infty T^{a,b}(M)\big)$ satisfies the product rule for any two tensor fields T_1, T_2:*

$$\nabla_X(T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes \nabla_X T_2.$$

Proof. Problem 22.4. □

22.6 The Second Bianchi Identity in Vector Form

If ∇ is a connection and $R(X,Y)$ is its curvature endomorphism on a manifold, then we define the ***Riemann curvature tensor*** or simply the ***curvature tensor*** to be

$$\operatorname{Rm}(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle, \quad X,Y,Z,W \in \mathfrak{X}(M).$$

The second Bianchi identity is obtained by differentiating the curvature. In vector form, this means taking the covariant derivative of the curvature tensor Rm.

Theorem 22.9 (The second Bianchi identity in vector form). *On a Riemannian manifold, if a connection ∇ is compatible with the metric, then for all X,Y,Z,V, $W \in \mathfrak{X}(M)$,*

$$\sum_{\text{cyclic in } X,Y,Z} (\nabla_X \operatorname{Rm})(Y,Z,V,W) = 0.$$

(This notation means we cyclically rotate X,Y,Z to obtain two other terms.)

Proof. Since $(\nabla_X \operatorname{Rm})(Y,Z,V,W)$ is $\mathcal{F}$-linear in all of its argument, it suffices to verify the identity at a point $p \in M$ and for X,Y,Z,V,W a frame of C^∞ vector fields near p. Such a frame is the coordinate frame $\partial_1,\dots,\partial_n$ of a normal coordinate neighborhood U of p. The advantage of using a normal coordinate system is that $\nabla_X Y = 0$ at p and $[X,Y] = 0$ on U for any coordinate vector fields X,Y, i.e., $X = \partial_i$, $Y = \partial_j$ for some i, j (Theorem 15.4). This greatly simplifies the expression for the covariant derivative of a tensor field.

Thus, if X,Y,Z,V,W are coordinate vector fields on a normal neighborhood of p, then at the point p

$$\begin{aligned}
(\nabla_X \operatorname{Rm})(Y,Z,V,W) &= X\big(\operatorname{Rm}(Y,Z,V,W)\big) - \operatorname{Rm}(\nabla_X Y,Z,V,W) - \cdots \\
&= X\big(\operatorname{Rm}(Y,Z,V,W)\big) \quad \text{since } \nabla_X Y(p) = 0 \\
&= X\langle R(Y,Z)V,W\rangle \\
&= \langle \nabla_X R(Y,Z)V,W\rangle + \langle R(Y,Z)V,\nabla_X W\rangle \\
&\qquad \text{by compatibility of } \nabla \text{ with the metric} \\
&= \langle \nabla_X\nabla_Y\nabla_Z V,W\rangle - \langle \nabla_X\nabla_Z\nabla_Y V,W\rangle.
\end{aligned}$$

Cyclically permuting X,Y,Z, we get

$$(\nabla_Y \operatorname{Rm})(Z,X,V,W) = \langle \nabla_Y\nabla_Z\nabla_X V,W\rangle - \langle \nabla_Y\nabla_X\nabla_Z V,W\rangle$$

and

$$(\nabla_Z \operatorname{Rm})(X,Y,V,W) = \langle \nabla_Z\nabla_X\nabla_Y V,W\rangle - \langle \nabla_Z\nabla_Y\nabla_X V,W\rangle.$$

Summing the three equations gives

$$\begin{aligned}
&(\nabla_X \operatorname{Rm})(Y,Z,V,W + (\nabla_Y \operatorname{Rm})(Z,X,V,W) + (\nabla_Z \operatorname{Rm})(X,Y,V,W \\
&\qquad = \langle R(X,Y)\nabla_Z V,W\rangle + \langle R(Y,Z)\nabla_X V,W\rangle + \langle R(Z,X)\nabla_Y V,W\rangle.
\end{aligned}$$

On the right-hand side, because $R(-,-)-$ is a tensor, we can evaluate the arguments at a single point p. Since $\nabla_X V$, $\nabla_Y V$, $\nabla_Z V$ all vanish at p, all three terms are zero. This establishes the second Bianchi identity in vector form. □

For the equivalence of the second Bianchi identity (Proposition 22.3) and the second Bianchi identity in vector form (Theorem 22.9), see [12, Vol. 1, Theorem 5.3, p. 155].

22.7 Ricci Curvature

The curvature tensor $\langle R(X,Y)Z,W\rangle$ is a complicated object, but from it one can construct other invariants of a Riemannian manifold M. One such is the sectional curvature introduced in Section 12.4.

Another is the ***Ricci curvature***, which associates to two tangent vectors $u,v \in T_pM$ at p the trace of the linear endomorphism

$$w \mapsto R(w,u)v.$$

Thus, if X,Y are C^∞ vector fields on an open set U in M and $E_1,\dots,E_n$ is an orthonormal frame on U, then

$$\begin{aligned}\operatorname{Ric}(X,Y) &= \operatorname{tr}\big(R(-,X)Y\big)\\ &= \sum_{i=1}^{n}\langle R(E_i,X)Y,E_i\rangle.\end{aligned}$$

By the symmetry properties of R, for the Riemannian connection

$$\begin{aligned}\operatorname{Ric}(X,Y) &= \sum\langle R(E_i,X)Y,E_i\rangle \quad \text{(by definition)}\\ &= \sum\langle R(Y,E_i)E_i,X\rangle \quad \text{(by Theorem 22.6)}\\ &= \sum\langle R(E_i,Y)X,E_i\rangle\\ &= \operatorname{Ric}(Y,X).\end{aligned}$$

Thus, the Ricci curvature $\operatorname{Ric}(X,Y)$ is a symmetric tensor of type $(0,2)$.

22.8 Scalar Curvature

At each point p of a Riemannian manifold M, the Ricci curvature is a symmetric bilinear form

$$\operatorname{Ric}: T_pM \times T_pM \to T_pM.$$

The ***scalar curvature*** at p is the ***trace*** of the Ricci curvature at p, defined as follows.

By the nondegeneracy of the Riemannian metric $\langle\ ,\ \rangle$, there is a unique linear map $\rho: T_pM \to T_pM$ such that

$$\operatorname{Ric}(u,v) = \langle \rho(u), v\rangle \quad \text{for all } u,v \in T_pM.$$

The scalar curvature $S(p)$ at the point p is defined to be the trace of ρ.

If $e_1,\dots,e_n$ is an orthonormal basis for T_pM, then

$$\begin{aligned}S(p) = \operatorname{tr}\rho &= \sum_j \langle \rho(e_j), e_j\rangle\\ &= \sum_j \operatorname{Ric}(e_j,e_j)\\ &= \sum_{i,j}\langle R(e_i,e_j)e_j,e_i\rangle.\end{aligned}$$

22.9 Defining a Connection Using Connection Matrices

In the next section we will describe how a connection on a vector bundle induces a connection on a pullback bundle. The easiest way to do this is to define a connection using connection matrices.

If ∇ is a connection on a vector bundle $E \to M$, then relative to each framed open set $(U,e_1,\dots,e_r)$ for E, there is a connection matrix ω_e. By Theorem 22.1, the connection matrix relative to another frame $\bar{e} = ea$ is related to ω_e by the formula

$$\omega_{\bar{e}} = a^{-1}\omega_e a + a^{-1}da. \tag{22.11}$$

For $s \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$, to define $\nabla_X s \in \Gamma(E)$, it suffices to define $(\nabla_X s)|_U$ for the open sets U of an open cover of M. By the definition of the restriction of a connection,

$$(\nabla_X s)|_U = \nabla_X^U(s|_U).$$

Thus, a connection ∇ on E is completely determined by its restrictions ∇^U to the open sets of an open cover for M. Conversely, a collection of connections ∇^U on the open sets U of a trivializing open cover of M that agree on pairwise intersections $U \cap U'$ defines a connection on E.

Equivalently, since a connection on a trivial bundle is completely specified by its connection matrix relative to a frame (see Section 11.2), a connection on a vector bundle can also be given by a collection of connection forms $\{\omega_e, (U,e)\}$ satisfying the compatibility conditions (22.11).

22.10 Induced Connection on a Pullback Bundle

In this section we show that under a C^∞ map $f\colon N \to M$, a connection on a vector bundle $E \to M$ pulls back to a unique connection on $f^*E \to N$.

Theorem 22.10. *Let ∇ be a connection on a vector bundle $E \to M$ and $f\colon N \to M$ a C^∞ map. Suppose M is covered by framed open sets $(U, e_1, \dots, e_r)$ for E and the connection matrix relative to the frame e is ω_e. Then there is a unique connection on f^*E whose connection matrix relative to the frame $f^*e_1, \dots, f^*e_r$ on $f^{-1}(U)$ is $f^*(\omega_e)$.*

Proof. The matrix $f^*(\omega_e)$ of 1-forms defines a connection on the trivial bundle $(f^*E)|_{f^{-1}(U)}$ relative to the frame f^*e (Section 11.2). If $\bar{e} = ea$ is another frame on U, then by Theorem 22.1

$$\omega_{\bar{e}} = a^{-1}\omega_e a + a^{-1}da \text{ on } U.$$

Taking the pullback under f^* gives

$$f^*(\omega_{\bar{e}}) = (f^*a)^{-1} f^*(\omega_e) f^*a + (f^*a)^{-1} d f^*a \text{ on } f^{-1}(U).$$

Since $f^*\bar{e} = (f^*e)(f^*a)$, the equation above shows that $\{f^*(\omega_e)\}$ satisfies the compatibility condition (22.11) and defines a unique connection on f^*E. □

Problems

22.1. Differential of the inverse of a matrix of functions

If $\omega = [\omega^i_j]$ is a matrix of differential forms on a manifold, we define $d\omega$ to be the matrix whose (i,j)-entry is $d(\omega^i_j)$. Let $a = [a^i_j]$ be a matrix of functions on a manifold. Prove that

$$d(a^{-1}) = -a^{-1}(da)a^{-1}.$$

(*Note*: Because a and da are matrices, we cannot combine the right-hand side into $-(da)a^{-2}$ as in calculus.)

22.2. Matrix of a linear transformation
Let $L\colon V \to V$ be a linear transformation of a finite dimensional vector space V. Suppose $e_1, \ldots, e_r$ and $\bar{e}_1, \ldots, \bar{e}_r$ are two bases of V such that $\bar{e}_j = \sum_i e_i a^i_j$. If $L(e_j) = \sum_i e_i \lambda^i_j$ and $L(\bar{e}_\ell) = \sum_k \bar{e}_k \bar{\lambda}^k_\ell$, prove that $\bar{\Lambda} = a^{-1}\Lambda a$, where $\bar{\Lambda} = [\bar{\lambda}^k_\ell]$ and $\Lambda = [\lambda^i_j]$ are the matrices of L with respect to the two bases $\bar{e}$ and e.

22.3. Generalized second Bianchi identity
Prove Proposition 22.4.

22.4. Product rule for the covariant derivative
Prove Theorem 22.8.

§23 Characteristic Classes

A connection on a vector bundle E of rank r over a manifold M can be represented locally by a matrix ω of 1-forms relative to a frame for E over an open set U. Similarly, the curvature of the connection can be represented by a matrix Ω of 2-forms over U. Under a change of frame $\bar{e} = ea$, the curvature matrix transforms by conjugation $\bar{\Omega} = a^{-1}\Omega a$. Thus, if $P(X)$ is a polynomial in r^2 variables invariant under conjugation by elements of $\mathrm{GL}(r,\mathbb{R})$, then the differential form $P(\Omega)$ will be independent of the frame and will define a global form on M. It turns out that this global form $P(\Omega)$ is closed and is independent of the connection. For a fixed vector bundle E, the cohomology class $[P(\Omega)]$ is therefore a well-defined element of $H^*(M)$ depending only on the invariant polynomial $P(X)$. This gives rise to an algebra homomorphism

$$c_E\colon\ \mathrm{Inv}\big(\mathfrak{gl}(r,\mathbb{R})\big) \to H^*(M),$$

called the ***Chern–Weil homomorphism*** from the algebra $\mathrm{Inv}\big(\mathfrak{gl}(r,\mathbb{R})\big)$ of invariant polynomials on $\mathfrak{gl}(r,\mathbb{R})$ to the de Rham cohomology algebra of M.

For each homogeneous invariant polynomial $P(X)$ of degree k, the cohomology class $[P(\Omega)] \in H^{2k}(M)$ is an isomorphism invariant of the vector bundle: if two vector bundles of rank r over M are isomorphic, then their cohomology classes associated to $P(X)$ are equal. In this sense, the class $[P(\Omega)]$ is characteristic of the vector bundle E and it is called a ***characteristic class*** of E. A global form $P(\Omega)$ representing a characteristic class is called a ***characteristic form***.

23.1 Invariant Polynomials on $\mathfrak{gl}(r,\mathbb{R})$

Let $X = [x^i_j]$ be an $r \times r$ matrix with indeterminate entries x^i_j. A ***polynomial*** $P(X)$ on $\mathfrak{gl}(r,\mathbb{R}) = \mathbb{R}^{r\times r}$ is a polynomial in the entries of X. A polynomial $P(X)$ on $\mathfrak{gl}(r,\mathbb{R})$ is said to be ***Ad*** $\mathrm{GL}(r,\mathbb{R})$***-invariant*** or simply ***invariant*** if

$$P(A^{-1}XA) = P(X) \quad \text{for all } A \in \mathrm{GL}(r,\mathbb{R}). \tag{23.1}$$

By Proposition B.1 in Appendix B, if (23.1) holds for all $r \times r$ matrices X of real numbers, then it holds when X is a matrix of indeterminates, i.e., $P(X)$ is an invariant polynomial on $\mathfrak{gl}(r,\mathbb{R})$. Thus, $\mathrm{tr}(X)$ and $\det(X)$ are examples of invariant polynomials on $\mathfrak{gl}(r,\mathbb{R})$.

Example 23.1 (*Coefficients of the characteristic polynomial*). Let $X = [x^i_j]$ be an $r \times r$ matrix of indeterminates and let λ be another indeterminate. The coefficients $f_k(X)$ of λ^{r-k} in

$$\det(\lambda I + X) = \lambda^r + f_1(X)\lambda^{r-1} + \cdots f_{r-1}(X)\lambda + f_r(X)$$

are polynomials on $\mathfrak{gl}(r,\mathbb{R})$. For any $A \in \mathrm{GL}(r,\mathbb{R})$ and any $r \times r$ matrix X of real numbers,

$$\det(\lambda I + A^{-1}XA) = \det(\lambda I + X).$$

Comparing coefficients of λ^{r-k}, we get

$$f_k(A^{-1}XA) = f_k(X) \quad \text{for all } A \in \mathrm{GL}(r,\mathbb{R}), X \in \mathfrak{gl}(r,\mathbb{R}).$$

By Proposition B.1, $f_1(X), \ldots, f_r(X)$ are all invariant polynomials on $\mathfrak{gl}(r,\mathbb{R})$. They are the ***coefficients of the characteristic polynomial*** of $-X$. (The characteristic polynomial of X is $\det(\lambda I - X)$.)

Example 23.2 (*Trace polynomials*). For any positive integer k, the polynomial $\Sigma_k(X) := \mathrm{tr}(X^k)$ is clearly invariant under conjugation for any $X \in \mathfrak{gl}(r,\mathbb{R})$. By Proposition B.1, $\Sigma_k(X)$ is an invariant polynomial on $\mathfrak{gl}(r,\mathbb{R})$. It is the kth ***trace polynomial***.

If $\mathcal{A}$ is any commutative $\mathbb{R}$-algebra with identity, then the canonical map $\mathbb{R} \to \mathcal{A}$ induces an algebra homomorphism $\mathbb{R}[x^i_j] \to \mathcal{A}[x^i_j]$. For any invariant polynomial $P(X)$ on $\mathfrak{gl}(r,\mathbb{R})$ and any invertible matrix $A \in \mathrm{GL}(r,\mathbb{R})$, the polynomial

$$P_A(X) = P(A^{-1}XA) - P(X)$$

is the zero polynomial in $\mathbb{R}[x^i_j]$. Under the homomorphism $\mathbb{R}[x^i_j] \to \mathcal{A}[x^i_j]$, the zero polynomial $P_A(X)$ maps to the zero polynomial in $\mathcal{A}[x^i_j]$. Thus, $P_A(X)$ is identically zero when evaluated on $\mathcal{A}^{r\times r}$. This means that for any invariant polynomial $P(X)$ on $\mathfrak{gl}(r,\mathbb{R})$ and $A \in \mathrm{GL}(r,\mathbb{R})$, the invariance condition

$$P(A^{-1}XA) = P(X)$$

holds not only for $r \times r$ matrices X of real numbers, but more generally for $r \times r$ matrices with entries in any $\mathbb{R}$-algebra $\mathcal{A}$ with identity.

23.2 The Chern–Weil Homomorphism

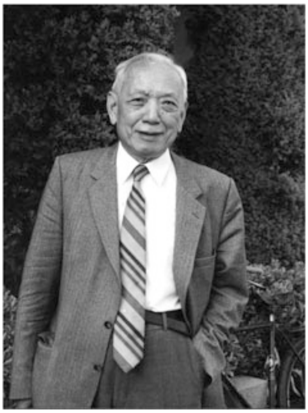
Let $E \to M$ be a C^∞ vector bundle over a manifold M. We assume that there is a connection ∇ on E; apart from this, there are no restrictions on E or M. Thus, we do not assume that E is a Riemannian bundle or that M is compact or orientable.

Let $\Omega = [\Omega^i_j]$ be the curvature matrix of ∇ relative to a frame $e = [e_1 \; \cdots \; e_r]$ on U. If $\bar{e} = [\bar{e}_1 \; \cdots \; \bar{e}_r]$ is another frame on U, then

$$\bar{e} = ea$$

for a C^∞ function $a\colon U \to \mathrm{GL}(r,\mathbb{R})$ and by Theorem 22.1, the curvature matrix $\bar{\Omega}$ relative to the frame $\bar{e}$ is

$$\bar{\Omega} = a^{-1}\Omega a.$$



Shiing-Shen Chern

(1911–2004)

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Let P be a homogeneous invariant polynomial of degree k on $\mathfrak{gl}(r,\mathbb{R})$. Fix a point $p \in U$. Let $\mathcal{A}$ be the commutative $\mathbb{R}$-algebra of covectors of even degrees at p,

$$\mathcal{A} = \bigoplus_{i=0}^{\infty} \bigwedge^{2i}(T_p^*M).$$

If $\Omega = [\Omega_j^i]$ is the curvature matrix of the connection ∇ relative to the frame e for E over U, then $\Omega_p \in \mathcal{A}^{r\times r}$ and

$$\bar{\Omega}_p = a(p)^{-1}\,\Omega_p\,a(p),$$

where $a(p) \in \mathrm{GL}(r,\mathbb{R})$. By the invariance of $P(X)$ under conjugation,

$$P(\bar{\Omega}_p) = P(a(p)^{-1}\Omega_p a(p)) = P(\Omega_p) \in \bigwedge^{2k}(T_p^*M).$$

As p varies over U,

$$P(\bar{\Omega}) = P(\Omega) \in \Omega^{2k}(U). \tag{23.2}$$

Thus, the $2k$-form $P(\Omega)$ on U is independent of the frame.

Let $\{U_\alpha\}_{\alpha\in\mathrm{A}}$ be a trivializing open cover for E. On each U_α, choose a frame $e^\alpha = [e_1^\alpha \cdots e_r^\alpha]$ for E, and let Ω_α be the curvature matrix of the connection ∇ relative to this frame. If $P(X)$ is a homogeneous invariant polynomial of degree k on $\mathfrak{gl}(r,\mathbb{R})$, then $P(\Omega_\alpha)$ is a $2k$-form on U_α. On the overlap $U_\alpha \cap U_\beta$, we have two $2k$-forms $P(\Omega_\alpha)$ and $P(\Omega_\beta)$. By (23.2) they are equal. Therefore, the collection of forms $\{P(\Omega_\alpha)\}_{\alpha\in A}$ piece together to give rise to a global $2k$-form on M, which we denote by $P(\Omega)$.

There are two fundamental results concerning the form $P(\Omega)$.



André Weil

(1906–1998)

Theorem 23.3. *Let E be a vector bundle of rank r on a manifold M, ∇ a connection on E, and P an invariant homogeneous polynomial of degree k on $\mathfrak{gl}(r,\mathbb{R})$. Then*

(i) *the global $2k$-form $P(\Omega)$ on M is closed;*
(ii) *the cohomology class of the closed form $P(\Omega)$ in $H^{2k}(M)$ is independent of the connection.*

For any vector bundle E of rank r over M and any connection ∇ on E, the map

$$c_E\colon \operatorname{Inv}(\mathfrak{gl}(r,\mathbb{R})) \to H^*(M),$$
$$P(X) \mapsto [P(\Omega)],$$

sends $P(X)Q(X)$ to $[P(\Omega)\wedge Q(\Omega)]$ and is clearly an algebra homomorphism. It is called the ***Chern–Weil homomorphism***.

23.3 Characteristic Forms Are Closed

In Examples 23.1 and 23.2 we introduced two sets of invariant polynomials on $\mathfrak{gl}(r,\mathbb{R})$:

(i) the coefficients $f_k(X)$ of the characteristic polynomial $\det(tI+X)$,
(ii) the trace polynomials $\Sigma_k(X) = \operatorname{tr}(X^k)$.

The polynomials $f_k(X)$ and the trace polynomials $\Sigma_k(X)$ are related by Newton's identity (Theorem B.14)

$$\Sigma_k - f_1\Sigma_{k-1} + f_2\Sigma_{k-2} - \cdots + (-1)^{k-1} f_{k-1}\Sigma_1 + (-1)^k k f_k = 0.$$

These two sets of polynomials play a crucial role in the theory of characteristic classes, because of the following algebraic theorem. A proof of this theorem can be found in Appendix B.

Theorem 23.4. *The ring* $\operatorname{Inv}\left(\mathfrak{gl}(r,\mathbb{R})\right)$ *of invariant polynomials on* $\mathfrak{gl}(r,\mathbb{R})$ *is generated as a ring either by the coefficients* $f_k(X)$ *of the characteristic polynomial or by the trace polynomials* Σ_k*:*

$$\operatorname{Inv}\left(\mathfrak{gl}(r,\mathbb{R})\right) = \mathbb{R}[f_1,\dots,f_r] = \mathbb{R}[\Sigma_1,\dots,\Sigma_r].$$

In particular, every invariant polynomial P on $\mathfrak{gl}(n,\mathbb{R})$ is a polynomial Q in $\Sigma_1,\dots,\Sigma_r$. Thus,

$$P(\Omega) = Q\big(\Sigma_1(\Omega),\dots,\Sigma_r(\Omega)\big).$$

This shows that it is enough to prove Theorem 23.3 for the trace polynomials, for if the $\Sigma_k(\Omega)$ are closed, then any polynomial in the $\Sigma_k(\Omega)$ is closed, and if the cohomology classes of the $\Sigma_k(\Omega)$ are independent of the connection ∇, then the same is true of any polynomial in the $\Sigma_k(\Omega)$.

Recall that if $A = [\alpha^i_j]$ and $B = [\beta^i_j]$ are matrices of C^∞ forms on a manifold with the number of columns of A equal to the number of rows of B, then by definition

(i) $(A\wedge B)^i_j = \sum_k \alpha^i_k \wedge \beta^k_i$,
(ii) $(dA)^i_j = d(\alpha^i_j)$.

The next proposition collects together some basic properties about matrices of differential forms, their wedge product, and their trace.

Proposition 23.5. *Let* $A = [\alpha^i_j]$ *and* $B = [\beta^i_j]$ *be matrices of* C^∞ *forms of degrees a and b, respectively, on a manifold.*

(i) *If* $A\wedge B$ *is defined, then*

$$(A\wedge B)^T = (-1)^{ab} B^T \wedge A^T.$$

(For matrices of functions, $(AB)^T = B^T A^T$*; for smooth forms,* $\alpha\wedge\beta = (-1)^{ab}\beta\wedge\alpha$*. Formula (i) generalizes both of these formulas.)*

(ii) *If $A \wedge B$ and $B \wedge A$ are both defined, then*

$$\operatorname{tr}(A \wedge B) = (-1)^{ab} \operatorname{tr}(B \wedge A).$$

(iii) *If $A = [a^i_j]$ is a square matrix of differential forms on M, then*

$$d \operatorname{tr} A = \operatorname{tr} dA.$$

Proof. (i) It suffices to compute the (i, j)-entry of both sides:

$$\left((A \wedge B)^T\right)^i_j = (A \wedge B)^j_i = \sum_k \alpha^j_k \wedge \beta^k_i,$$

$$\left(B^T \wedge A^T\right)^i_j = \sum_k (B^T)^i_k \wedge (A^T)^k_j = \sum_k \beta^k_i \wedge \alpha^j_k.$$

Since α^j_k is an a-form and β^k_i is a b-form, the proposition follows.

(ii)

$$\operatorname{tr}(A \wedge B) = \sum_i (A \wedge B)^i_i = \sum_i \sum_k \alpha^i_k \wedge \beta^k_i,$$

$$\operatorname{tr}(B \wedge A) = \sum_k (B \wedge A)^k_k = \sum_k \sum_i \beta^k_i \wedge \alpha^i_k.$$

(iii)

$$d \operatorname{tr} A = d\left(\sum_i a^i_i\right) = \sum_i da^i_i = \operatorname{tr} dA. \qquad \square$$

Proof (*of Theorem 23.3*(i) *for trace polynomials*). Let ω and Ω be the connection and curvature matrices relative to a frame e on U. Then

$$\begin{aligned}
d \operatorname{tr}(\Omega^k) &= \operatorname{tr} d(\Omega^k) && \big(\text{Proposition 23.5 (iii)}\big)\\
&= \operatorname{tr}(\Omega^k \wedge \omega - \omega \wedge \Omega^k) && (\text{Generalized second Bianchi identity})\\
&= \operatorname{tr}(\Omega^k \wedge \omega) - \operatorname{tr}(\omega \wedge \Omega^k) &&\\
&= 0 && \big(\text{Proposition 23.5 (ii)}\big) \qquad \square
\end{aligned}$$

23.4 Differential Forms Depending on a Real Parameter

If ω_t is a C^∞ k-form on a manifold M depending on a real parameter $t \in J$ for some open interval $J \subset \mathbb{R}$, then locally, on a chart $(U, x^1, \ldots, x^n)$ for the manifold, ω_t is a linear combination

$$\omega_t = \sum_I a_I(x,t)\, dx^I,$$

where $I = (i_1 < \cdots < i_k)$ is a multi-index. We say that ω_t ***depends smoothly on*** t or ***varies smoothly with*** t if for every U in an atlas $\{(U, x^1, \ldots, x^n)\}$ for M, the coefficients $a_I(x,t)$ are all C^∞ functions on $U \times J$. In this case, we also say that $\{\omega_t\}_{t \in J}$ is a ***smoothly varying family*** of C^∞ k-forms on M.

Suppose $\{\omega_t\}_{t\in J}$ is such a family. For each $p \in M$, the map $t \mapsto \omega_{t,p}$ is a map from J to the vector space $\bigwedge^k(T_p^*M)$. We define $d\omega_t/dt$, also written $\dot{\omega}_t$, to be the k-form on M such that at each $p \in M$,

$$\left(\frac{d\omega_t}{dt}\right)_p = \frac{d}{dt}\omega_{t,p}.$$

This definition makes sense because the vector space $\bigwedge^k(T_p^*M)$ is finite-dimensional. Locally,

$$\frac{d}{dt}\omega_t = \sum_I \frac{\partial a_I}{\partial t}(x,t)\,dx^I.$$

Similarly, for $a < b$ in J, we define $\int_a^b \omega_t\,dt$ to be the k-form on M such that at each $p \in M$,

$$\left(\int_a^b \omega_t\,dt\right)_p = \int_a^b \omega_{t,p}\,dt.$$

Locally,

$$\int_a^b \omega_t\,dt = \sum_I \left(\int_a^b a_I(x,t)\,dt\right) dx^I. \tag{23.3}$$

All of these notions — C^∞ dependence on t and differentiation and integration with respect to t — extend entry by entry to a matrix of forms depending on a real parameter t.

Proposition 23.6. *Suppose $\omega = \omega_t$ and $\tau = \tau_t$ are matrices of C^∞ differential forms on a manifold M depending smoothly on a real parameter t. Assume that the wedge product $\omega \wedge \tau$ makes sense, i.e., the number of columns of ω is equal to the number of rows of τ.*

(i) *If $\omega = [\omega_j^i(t)]$ is a square matrix, then*

$$\frac{d}{dt}(\operatorname{tr}\omega) = \operatorname{tr}\left(\frac{d\omega}{dt}\right).$$

(ii) *(Product rule)*

$$\frac{d}{dt}(\omega \wedge \tau) = \dot{\omega} \wedge \tau + \omega \wedge \dot{\tau}.$$

(iii) *(Commutativity of d/dt with d)* $\dfrac{d}{dt}(d\omega_t) = d\left(\dfrac{d}{dt}\omega_t\right)$.

(iv) *(Commutativity of integration with respect to t with d) If $[a,b] \subset J$, then*

$$\int_a^b d\omega_t\,dt = d\left(\int_a^b \omega_t\,dt\right).$$

Proof. (i)

$$\frac{d}{dt}(\operatorname{tr}\omega) = \frac{d}{dt}\sum \omega_i^i(t) = \sum \frac{d\omega_i^i}{dt} = \operatorname{tr}\left(\frac{d\omega}{dt}\right).$$

The proofs of (ii) and (iii) are straightforward. We leave them as exercises (Problems 23.1 and 23.2).

Since both sides of (iv) are matrices of differential forms on M, it suffices to prove the equality locally and for each entry. On a coordinate open set $(U, x^1, \dots, x^n)$,

$$\omega_t = \sum_I a_I(x,t)\, dx^I \quad \text{and} \quad d\omega_t = \sum_{i,I} \frac{\partial a_I}{\partial x^i}\, dx^i \wedge dx^I.$$

By (23.3),

$$\int_a^b (d\omega_t)\, dt = \sum_{i,I}\left(\int_a^b \frac{\partial a_I}{\partial x^i}\, dt\right) dx^i \wedge dx^I. \tag{23.4}$$

By (23.3) and the definition of the exterior derivative on U,

$$\begin{aligned} d\int_a^b \omega_t\, dt &= \sum_I d\left(\left(\int_a^b a_I(x,t)\, dt\right) dx^I\right) \\ &= \sum_{I,i} \frac{\partial}{\partial x^i}\left(\int_a^b a_I(x,t)\, dt\right) dx^i \wedge dx^I. \end{aligned} \tag{23.5}$$

Comparing (23.5) and (23.4), we see that the proposition is equivalent to differentiation under the integral sign:

$$\frac{\partial}{\partial x^i}\int_a^b a_I(x,t)\, dt = \int_a^b \frac{\partial}{\partial x^i} a_I(x,t)\, dt.$$

By a theorem of real analysis [15, §9.7, Proposition, p. 517], this is always possible for an integral over a finite interval $[a,b]$ provided that a_I and $\partial a_I/\partial x^i$ are continuous on $U \times [a,b]$. □

23.5 Independence of Characteristic Classes of a Connection

Recall from Proposition 10.5 that a convex linear combination of connections on a vector bundle is again a connection. Given two connections ∇^0 and ∇^1 on a smooth vector bundle $\pi\colon E \to M$, we define for each $t \in \mathbb{R}$ the convex linear combination

$$\nabla^t = (1-t)\nabla^0 + t\nabla^1,$$

and then ∇^t is a connection on E.

Let ω_t and Ω_t be the connection and curvature matrices of ∇^t relative to a frame $e_1, \dots, e_r$ for E over an open set U. For $X \in \mathfrak{X}(U)$,

$$\begin{aligned} \nabla_X^t e_j &= (1-t)\nabla_X^0 e_j + t\nabla_X^1 e_j \\ &= \sum_i \left((1-t)(\omega_0)_j^i + t(\omega_1)_j^i\right)(X) e_i. \end{aligned}$$

Hence,

$$\omega_t = (1-t)\omega_0 + t\omega_1.$$

This proves that the connection matrix ω_t depends smoothly on t. By the second structural equation, the curvature matrix Ω_t also depends smoothly on t.

To show that the cohomology class of $\operatorname{tr}(\Omega_t^k)$ is independent of t, it is natural to differentiate $\operatorname{tr}(\Omega_t^k)$ with respect to t and hope that the derivative will be a global exact form on M. In the following proof, in order to simplify the notation, we often suppress the wedge product.

Proposition 23.7. *If ∇^t is a family of connections on a vector bundle E whose connection and curvature matrices ω_t and Ω_t on a framed open set U depend smoothly on a real parameter t, then*

$$\frac{d}{dt}(\operatorname{tr}\Omega^k) = d(k\operatorname{tr}(\Omega^{k-1}\dot{\omega})),$$

where $\Omega = \Omega_t$.

Proof.

$$\begin{aligned}
\frac{d}{dt}(\operatorname{tr}\Omega^k) &= \operatorname{tr}\left(\frac{d}{dt}\Omega^k\right) && \text{(Proposition 23.6 (i))}\\
&= \operatorname{tr}(\dot{\Omega}\Omega^{k-1} + \Omega\dot{\Omega}\Omega^{k-2} + \cdots + \Omega^{k-1}\dot{\Omega}) && \text{(product rule)}\\
&= k\operatorname{tr}(\Omega^{k-1}\dot{\Omega}) && \big(\text{Proposition 23.5 (ii)}\big)
\end{aligned}$$

To show that $\operatorname{tr}(\Omega^{k-1}\dot{\Omega})$ is an exact form, we differentiate the structural equation $\Omega = d\omega + \omega\wedge\omega$ with respect to t:

$$\begin{aligned}
\dot{\Omega} &= \frac{d}{dt}d\omega + \frac{d}{dt}(\omega\wedge\omega)\\
&= d\dot{\omega} + \dot{\omega}\wedge\omega + \omega\wedge\dot{\omega},
\end{aligned}$$

by the commutativity of d/dt with d (Proposition 23.6(iii)) and the product rule (Proposition 23.6(ii)). Hence,

$$\begin{aligned}
\operatorname{tr}(\Omega^{k-1}\dot{\Omega}) &= \operatorname{tr}(\Omega^{k-1}d\dot{\omega} + \Omega^{k-1}\dot{\omega}\omega + \Omega^{k-1}\omega\dot{\omega})\\
&= \operatorname{tr}(\Omega^{k-1}d\dot{\omega} - \omega\Omega^{k-1}\dot{\omega} + \Omega^{k-1}\omega\dot{\omega})\\
&\qquad \text{(by Proposition 23.5(ii), there is a sign change}\\
&\qquad \text{because both } \Omega^{k-1}\dot{\omega} \text{ and } \omega \text{ have odd degree)}\\
&= \operatorname{tr}\left(\Omega^{k-1}d\dot{\omega} + (d\Omega^{k-1})\dot{\omega}\right)\\
&\qquad \text{(generalized second Bianchi identity)}\\
&= \operatorname{tr}\left(d(\Omega^{k-1}\dot{\omega})\right) = d\left(\operatorname{tr}(\Omega^{k-1}\dot{\omega})\right) \quad \text{(Proposition 23.5 (iii)).}
\end{aligned}$$

□

Remark 23.8. Under a change of frame $\bar{e} = ea$,

$$\bar{\omega}_t = a^{-1}\omega_t a + a^{-1}da.$$

Differentiating with respect to t kills $a^{-1}da$, since a does not depend on t, hence

$$\dot{\bar{\omega}} = a^{-1}\dot{\omega}a.$$

Thus,

$$\operatorname{tr}(\bar{\Omega}^{k-1}\dot{\bar{\omega}}) = \operatorname{tr}(a^{-1}\Omega^{k-1}aa^{-1}\dot{\omega}a) = \operatorname{tr}(a^{-1}\Omega^{k-1}\dot{\omega}a) = \operatorname{tr}(\Omega^{k-1}\dot{\omega}).$$

This shows that $\operatorname{tr}(\Omega^{k-1}\dot{\omega})$ is independent of the frame and so can be pieced together into a global form on M.

Proof (*of Theorem 23.3*(ii)). Suppose ∇^0 and ∇^1 are two connections on the vector bundle E. Define a family of connections

$$\nabla^t = (1-t)\nabla^0 + t\nabla^1, \; t \in \mathbb{R},$$

as before, with connection matrix ω_t and curvature matrix Ω_t on a framed open set. By Proposition 23.7,

$$\frac{d}{dt}(\operatorname{tr}\Omega_t^k) = d(k\operatorname{tr}(\Omega_t^{k-1}\dot{\omega}_t)).$$

Integrating both sides with respect to t from 0 to 1 gives

$$\int_0^1 \frac{d}{dt}(\operatorname{tr}\Omega_t^k)\,dt = \operatorname{tr}(\Omega_t^k)\Big]_0^1 = \operatorname{tr}(\Omega_1^k) - \operatorname{tr}(\Omega_0^k)$$

and

$$\int_0^1 d(k\operatorname{tr}(\Omega^{k-1}\dot{\omega}))\,dt = d\int_0^1 k\operatorname{tr}(\Omega^{k-1}\dot{\omega})\,dt$$

by Proposition 23.6(iv).

Thus,

$$\operatorname{tr}(\Omega_1^k) - \operatorname{tr}(\Omega_0^k) = d\int_0^1 k\operatorname{tr}(\Omega^{k-1}\dot{\omega})\,dt$$

In this equation, both $\operatorname{tr}(\Omega_1^k)$ and $\operatorname{tr}(\Omega_0^k)$ are global forms on M. As noted in Remark 23.8, so is $\operatorname{tr}(\Omega^{k-1}\dot{\omega})$. Passing to cohomology classes, we get $[\operatorname{tr}(\Omega_1^k)] = [\operatorname{tr}(\Omega_0^k)]$. This proves that the cohomology class of $\operatorname{tr}(\Omega^k)$ is independent of the connection. □

For any invariant homogeneous polynomial $P(X)$ of degree k on $\mathfrak{gl}(r,\mathbb{R})$, since $P(\Omega)$ is a polynomial in $\operatorname{tr}(\Omega)$, $\operatorname{tr}(\Omega^2)$, $\dots$, $\operatorname{tr}(\Omega^k)$, the characteristic form $P(\Omega)$ is closed and the cohomology class of $P(\Omega)$ is independent of the connection.

23.6 Functorial Definition of a Characteristic Class

A ***characteristic class*** on real vector bundles associates to each manifold M a map

$$c_M\colon \begin{Bmatrix}\text{isomorphism classes of real}\\ \text{vector bundles over } M\end{Bmatrix} \to H^*(M).$$

such that if $f\colon N \to M$ is a map of C^∞ manifolds and $E \to M$ is a vector bundle over M, then

$$c_N(f^*E) = f^*c_M(E). \tag{23.6}$$

Let $\operatorname{Vect}_k(M)$ be the set of all isomorphism classes of rank k vector bundles over M. Both $\operatorname{Vect}_k(\ )$ and $H^*(\ )$ are functors on the category of smooth manifolds. In functorial language a ***characteristic class*** is a natural transformation $c\colon \operatorname{Vect}_k(\ ) \to H^*(\ )$. By the definition of a natural transformation, if $f\colon N \to M$ is a C^∞ map of manifolds, then there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Vect}_k(N) & \xrightarrow{c_N} & H^*(N) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \operatorname{Vect}_k(M) & \xrightarrow[c_M]{} & H^*(M), \end{array}$$

which is precisely (23.6). For this reason, property (23.6) is often called the ***naturality*** property of the characteristic class c.

23.7 Naturality

We will now show that if $E \to M$ is a vector bundle of rank r and P is an $\operatorname{Ad}\big(\mathrm{GL}(r,\mathbb{R})\big)$-invariant polynomial on $\mathfrak{gl}(r,\mathbb{R})$, then the cohomology class $[P(\Omega)]$ satisfies the naturality property and therefore defines a characteristic class.

If ∇ is a connection on E, with connection matrices ω_e relative to frames e for E, by Theorem 22.10 there is a unique connection ∇' on $f^*E \to N$ whose connection matrix relative to the frame f^*e is $f^*(\omega_e)$.

The induced curvature form on f^*E is therefore

$$f^*\omega_e + \frac{1}{2}[f^*\omega_e, f^*\omega_e] = f^*\Omega_e.$$

Since f^* is an algebra homomorphism,

$$P(f^*\Omega_e) = f^*P(\Omega_e),$$

which proves the naturality of the characteristic class $[P(\Omega)]$.

Problems

23.1. Product rule for forms depending on t

(a) Suppose $\omega = \omega_t$ and $\tau = \tau_t$ are C^∞ differential forms on M depending smoothly on a real parameter t. Denote the derivative of ω by $d\omega/dt$ or $\dot{\omega}$. Prove the product rule

$$\frac{d}{dt}(\omega \wedge \tau) = \dot{\omega} \wedge \tau + \omega \wedge \dot{\tau}.$$

(b) Prove the same formula if ω and τ are matrices of k-forms and ℓ-forms on M depending smoothly on t.

(c) If α, β, γ are matrices of differential forms on M depending smoothly on $t \in J$ for which $\alpha \wedge \beta \wedge \gamma$ is defined, prove that

$$\frac{d}{dt}(\alpha\beta\gamma) = \dot{\alpha}\beta\gamma + \alpha\dot{\beta}\gamma + \alpha\beta\dot{\gamma}.$$

23.2. Commutativity of d/dt with the exterior derivative

Suppose ω_t is a C^∞ k-form on a manifold M depending smoothly on t in some open interval $J \subset \mathbb{R}$. Prove that

$$\frac{d}{dt}(d\omega_t) = d(\dot{\omega}_t).$$

(*Hint*: Write out ω_t in terms of local coordinates $x^1, \ldots, x^n$ near p.)

§24 Pontrjagin Classes

Suppose E is a smooth vector bundle of rank r over a manifold M. In the preceding section we showed that starting with any connection ∇ on E, if $P(X)$ is a homogeneous invariant polynomial of degree k on $\mathfrak{gl}(r,\mathbb{R})$ and Ω is the curvature matrix of ∇ relative to any frame for E, then $P(\Omega)$ defines a closed global $2k$-form on M, whose cohomology class is independent of the connection. The cohomology class $[P(\Omega)]$ is called the ***characteristic class*** associated to P of the vector bundle E. This construction gives rise to the Chern–Weil homomorphism,

$$c_E\colon \operatorname{Inv}\big(\mathfrak{gl}(r,\mathbb{R})\big) \to H^*(M)$$

from the algebra of invariant polynomials on $\mathfrak{gl}(r,\mathbb{R})$ to the de Rham cohomology algebra of M. The image of the Chern–Weil homomorphism is precisely the subalgebra of characteristic classes of E associated to invariant polynomials.



Lev Pontrjagin

(1908–1988)

Since the algebra $\operatorname{Inv}\big(\mathfrak{gl}(r,\mathbb{R})\big)$ of invariant polynomials on $\mathfrak{gl}(r,\mathbb{R})$ is generated by the coefficients $f_k(X)$ of the characteristic polynomials and by the trace polynomials $\operatorname{tr}(X^k)$ (Appendix B, Theorem B.17), to determine all the characteristic classes of E, it suffices to calculate the characteristic classes associated to either set of polynomials. The characteristic classes arising from the $f_k(X)$ are called ***Pontrjagin classes***.

24.1 Vanishing of Characteristic Classes

In Section 11.4, we learned that on a Riemannian bundle E over a manifold M the curvature matrix of a metric connection relative to an orthonormal frame is skew-symmetric:

$$\Omega = \begin{bmatrix} 0 & \Omega_2^1 & \cdots & \Omega_r^1 \\ -\Omega_2^1 & 0 & \cdots & \\ \vdots & & \ddots & \vdots \\ -\Omega_r^1 & & \cdots & 0 \end{bmatrix}.$$

From the skew-symmetry of the curvature matrix, we see that $f_1(\Omega) = \operatorname{tr}(\Omega) = 0$.

This example is an instance of a general phenomenon: if k is odd, then the cohomology class $[\mathrm{tr}(\Omega^k)] = 0$ for any connection on any vector bundle. To see this, we begin with two simple algebraic lemmas.

Lemma 24.1. *If A is a skew-symmetric matrix of 2-forms on a manifold, then A^k is symmetric for k even and skew-symmetric for k odd.*

Proof. By Proposition 23.5(i),

$$(A \wedge A)^T = A^T \wedge A^T = (-A) \wedge (-A) = A \wedge A$$

and

$$((A \wedge A) \wedge A)^T = A^T \wedge (A^T \wedge A^T) = (-A) \wedge (-A) \wedge (-A) = -A \wedge A \wedge A.$$

Hence, A^2 is symmetric and A^3 is skew-symmetric. The general case proceeds by induction. □

Lemma 24.2. *Suppose 2 is not a zero divisor in a ring R. Then the diagonal entries of a skew-symmetric matrix over R are all zero.*

Proof. A skew-symmetric matrix is a square matrix $A = [a^i_j]$ such that $A^T = -A$. This implies that $a^i_i = -a^i_i$. Hence, $2a^i_i = 0$. Since 2 is not a zero divisor in R, one has $a^i_i = 0$ for all i. □

Next, we recall a few facts about Riemannian metrics, connections, and curvature on a vector bundle.

(i) It is possible to construct a Riemannian metric on any smooth vector bundle (Theorem 10.8).
(ii) On any Riemannian bundle there is a metric connection, a connection compatible with the metric (Proposition 10.12).
(iii) Relative to an orthonormal frame, the curvature matrix Ω of a metric connection is skew-symmetric (Propositions 11.4 and 11.5).

By Lemma 24.1, if Ω is the curvature matrix of a metric connection relative to an orthonormal frame, then for an odd k the matrix Ω^k is skew-symmetric. Since the diagonal entries of a skew-symmetric matrix are all zero, $\mathrm{tr}(\Omega^k) = 0$ for all odd k.

Even if the frame $\bar{e}$ is not orthonormal, the same result holds for the curvature matrix of a metric connection. This is because the curvature matrix relative to $\bar{e}$ is $\bar{\Omega} = a^{-1}\Omega a$, where a is the change of basis matrix defined by $\bar{e} = ea$. Since the trace polynomial $\mathrm{tr}(X^k)$ is invariant under conjugation, if Ω is the curvature matrix relative to an orthonormal frame e, then

$$\mathrm{tr}(\bar{\Omega}^k) = \mathrm{tr}(a^{-1}\Omega^k a) = \mathrm{tr}(\Omega^k) = 0 \text{ for } k \text{ odd.}$$

If the connection ∇ on E is not compatible with a metric on E, then $\mathrm{tr}(\Omega^k)$ need not be zero. However, by Theorem 23.3(ii) the cohomology class $[\mathrm{tr}(\Omega^k)]$ is still zero, because the cohomology class is independent of the connection.

Example. Let E be any smooth vector bundle over M. Put a Riemannian metric on E and let ∇ be a metric connection on E, with curvature matrix Ω on a framed open set. Suppose $P(X)$ is a homogeneous invariant polynomial of degree 3 on $\mathfrak{gl}(r,\mathbb{R})$. Since the trace polynomials Σ_1, Σ_2, $\Sigma_3, \dots$ generate the ring of invariant polynomials, P is a linear combination of monomials of degree 3 in the trace polynomials. There are only three such monomials, Σ_1^3, $\Sigma_1\Sigma_2$ and Σ_3. Hence,

$$P(X) = a\Sigma_1(X)^3 + b\Sigma_1(X)\Sigma_2(X) + c\Sigma_3(X)$$

for some constants $a,b,c \in \mathbb{R}$. Since the odd trace polynomials $\Sigma_1(\Omega)$ and $\Sigma_3(\Omega)$ are both zero, $P(\Omega) = 0$. Thus, the characteristic class of a vector bundle associated to any homogeneous invariant polynomial of degree 3 is zero.

Clearly, there is nothing special about degree 3 except that it is odd.

Theorem 24.3. *If a homogeneous invariant polynomial $P(X)$ on $\mathfrak{gl}(r,\mathbb{R})$ has odd degree k, then for any connection ∇ on any vector bundle $E \to M$ with curvature matrix Ω, the cohomology class $[P(\Omega)]$ is zero in $H^{2k}(M)$.*

Proof. Put a Riemannian metric on E and let ∇' be a metric connection on E, with curvature matrix Ω' on a framed open set. By Theorem 23.4, $P(X)$ is a linear combination with real coefficients of monomials $\Sigma_1^{i_1}\cdots\Sigma_r^{i_r}$ in the trace polynomials. Since k is odd, every monomial term of $P(X)$ has odd degree and so must contain a trace polynomial Σ_j of odd degree j as a factor. The vanishing of the odd trace polynomials $\Sigma_j(\Omega')$ implies that $P(\Omega') = 0$. By Theorem 23.3(ii), if ∇ is any connection on E with curvature matrix Ω, then $[P(\Omega)] = [P(\Omega')] = 0$. □

24.2 Pontrjagin Classes

To calculate the characteristic classes of a vector bundle $E \to M$, we can always use a metric connection ∇ compatible with some Riemannian metric on E. Let Ω be the curvature matrix of such a connection ∇ on a framed open set. From the proof of Theorem 24.3, we see that for k odd, the polynomials $\operatorname{tr}(\Omega^k)$ and $f_k(\Omega)$ are all zero. Thus, the ring of characteristic classes of E has two sets of generators:

(i) the trace polynomials of even degrees

$$[\operatorname{tr}(\Omega^2)], [\operatorname{tr}(\Omega^4)], [\operatorname{tr}(\Omega^6)], \dots$$

and

(ii) the coefficients of even degrees of the characteristic polynomial $\det(\lambda I + \Omega)$

$$[f_2(\Omega)], [f_4(\Omega)], [f_6(\Omega)], \dots.$$

Definition 24.4. The ***kth Pontrjagin class*** $p_k(E)$ of a vector bundle E over M is defined to be

$$p_k(E) = \left[f_{2k}\left(\frac{\mathrm{i}}{2\pi}\Omega\right)\right] \in H^{4k}(M). \tag{24.1}$$

In this definition the factor of i is to ensure that other formulas will be sign-free. Since f_{2k} is homogeneous of degree $2k$, the purely imaginary number i disappears in $f_{2k}\big((\mathrm{i}/2\pi)\Omega\big)$. A closed differential q-form on M is said to be ***integral*** if it gives an integer when integrated over any compact oriented submanifold of dimension q of M. The factor of $1/2\pi$ in the definition of the Pontrjagin class in (24.1) ensures that $p_k(E)$ is the class of an integral form. If the rank of E is r, then Ω is an $r \times r$ matrix of 2-forms. For any real number $x \in \mathbb{R}$, let $\lfloor x \rfloor$ be the greatest integer $\le x$. Then one can collect together all the Pontrjagin classes in a single formula:

$$\begin{aligned}\det\left(\lambda I + \frac{\mathrm{i}}{2\pi}\Omega\right) &= \lambda^r + f_1\left(\frac{\mathrm{i}}{2\pi}\Omega\right)\lambda^{r-1} + f_2\left(\frac{\mathrm{i}}{2\pi}\Omega\right)\lambda^{r-2} + \cdots + f_r\left(\frac{\mathrm{i}}{2\pi}\Omega\right)\\ &= \lambda^r + p_1\lambda^{r-2} + \cdots + p_{\lfloor \frac{r}{2}\rfloor}\lambda^{r-2\lfloor\frac{r}{2}\rfloor},\end{aligned}$$

where r is the rank of the vector bundle E. Setting $\lambda = 1$, we get

$$\det\left(I + \frac{\mathrm{i}}{2\pi}\Omega\right) = 1 + p_1 + \cdots + p_{\lfloor\frac{r}{2}\rfloor}.$$

We call this expression the ***total Pontrjagin class*** of E and denote it by $p(E)$. Note that $p_k = f_{2k}\left(\frac{\mathrm{i}}{2\pi}\Omega\right) \in H^{4k}(M)$.

Let E be a vector bundle of rank r over a compact oriented manifold M of dimension $4m$. A monomial $p_1^{a_1}p_2^{a_2}\cdots p_{\lfloor r/2\rfloor}^{a_{\lfloor r/2\rfloor}}$ of weighted degree

$$4\left(a_1 + 2a_2 + \cdots + \left\lfloor\frac{r}{2}\right\rfloor a_{\lfloor\frac{r}{2}\rfloor}\right) = 4m$$

represents a cohomology class of degree $4m$ on M and can be integrated over M; the resulting number $\int_M p_1^{a_1}p_2^{a_2}\cdots p_{\lfloor r/2\rfloor}^{a_{\lfloor r/2\rfloor}}$ is called a ***Pontrjagin number*** of E. The ***Pontrjagin numbers*** of a manifold M of dimension $4m$ are defined to be the Pontrjagin numbers of its tangent bundle TM.

Example 24.5. For a compact oriented manifold M of dimension 4, the only Pontrjagin number is $\int_M p_1$. For a compact oriented manifold of dimension 8, there are two Pontrjagin numbers $\int_M p_1^2$ and $\int_M p_2$.

24.3 The Whitney Product Formula

The total Pontrjagin class of a direct sum of vector bundles can be easily computed using the Whitney product formula.

Theorem 24.6 (Whitney product formula). *If E' and E'' are vector bundles over M, then*

$$p(E' \oplus E'') = p(E')p(E'').$$

Proof. Let ∇' and ∇'' be connections on the vector bundles E' and E'', respectively. We define a connection $\nabla := \nabla' \oplus \nabla''$ on $E := E' \oplus E''$ by

$$\nabla_X \begin{bmatrix} s' \\ s'' \end{bmatrix} = \begin{bmatrix} \nabla'_X s' \\ \nabla''_X s'' \end{bmatrix} \text{ for } X \in \mathfrak{X}(M), s' \in \Gamma(E'), s'' \in \Gamma(E'').$$

For $f \in C^\infty(M)$,

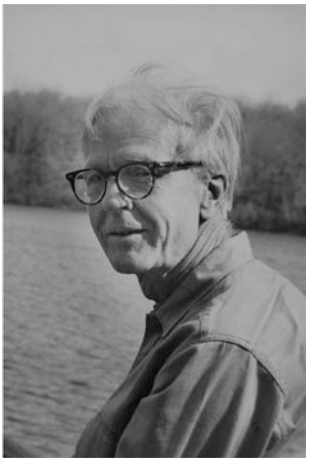
$$\nabla_{fX} \begin{bmatrix} s' \\ s'' \end{bmatrix} = \begin{bmatrix} \nabla'_{fX} s' \\ \nabla''_{fX} s'' \end{bmatrix} = f\nabla_X \begin{bmatrix} s' \\ s'' \end{bmatrix}$$

and

$$\nabla_X \begin{bmatrix} fs' \\ fs'' \end{bmatrix} = (Xf) \begin{bmatrix} s' \\ s'' \end{bmatrix} + f\nabla''_X \begin{bmatrix} s' \\ s'' \end{bmatrix}.$$

Therefore, ∇ is a connection on E.

Let $e' = [e'_1 \ \cdots \ e'_r]$ and $e'' = [e''_1 \ \cdots \ e''_s]$ be frames for E' and E'', respectively, over an open set U. Then $e = [e' \ e'']$ is a frame of $E' \oplus E''$ over U. Suppose the connection matrices of ∇' and ∇'' relative to e' and e'' are ω' and ω'' and the curvature matrices are Ω' and Ω'', respectively. Then the connection matrix of ∇ relative to the frame e is



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(1907–1989)

$$\omega = \begin{bmatrix} \omega' & 0 \\ 0 & \omega'' \end{bmatrix}$$

and the curvature matrix is

$$\begin{aligned} \Omega = d\omega + \omega \wedge \omega &= \begin{bmatrix} d\omega' & 0 \\ 0 & d\omega'' \end{bmatrix} + \begin{bmatrix} \omega' \wedge \omega' & 0 \\ 0 & \omega'' \wedge \omega'' \end{bmatrix} \\ &= \begin{bmatrix} \Omega' & 0 \\ 0 & \Omega'' \end{bmatrix}. \end{aligned}$$

Therefore, the total Pontrjagin class of $E' \oplus E''$ is

$$\begin{aligned} p(E' \oplus E'') &= \det\left(I + \frac{\mathrm{i}}{2\pi} \begin{bmatrix} \Omega' & 0 \\ 0 & \Omega'' \end{bmatrix} \right) \\ &= \det \begin{bmatrix} I + \frac{\mathrm{i}}{2\pi}\Omega' & 0 \\ 0 & I + \frac{\mathrm{i}}{2\pi}\Omega'' \end{bmatrix} \\ &= \det\left(I + \frac{\mathrm{i}}{2\pi}\Omega' \right) \det\left(I + \frac{\mathrm{i}}{2\pi}\Omega'' \right) \\ &= p(E')p(E''). \end{aligned}$$

□

§25 The Euler Class and Chern Classes

For dimension reasons alone, we cannot expect the Pontrjagin classes to give a generalized Gauss–Bonnet theorem. A Pontrjagin number makes sense only on a compact oriented manifold of dimension $4m$, whereas a generalized Gauss–Bonnet theorem should hold for a compact oriented manifold of any even dimension.

We used a Riemannian structure on a vector bundle E to show that the characteristic classes defined by $\mathrm{Ad}\big(\mathrm{GL}(r,\mathbb{R})\big)$-invariant polynomials of odd degree all vanish, and we used the orientability of the manifold M to integrate a top form, but we have not used the orientability assumption on the bundle E. This is the final ingredient for a generalized Gauss–Bonnet theorem.

A C^∞ ***complex vector bundle*** is a smooth map $\pi\colon E \to M$ of manifolds that is locally of the form $U \times \mathbb{C}^r \to U$. Just as complex numbers are necessary in the theory of polynomial equations, so complex vector bundles play an essential role in many areas of geometry. Examples abound: the tangent bundle of a complex manifold, the tautological subbundle on a complex Grassmannian, the hyperplane bundle of a complex projective variety.

Most of the concepts we developed for real vector bundles have obvious counterparts for complex vector bundles. In particular, the analogues of Pontrjagin classes for complex vector bundles are the Chern classes.

In this section we define the Euler class of a real oriented vector bundle of even rank and the Chern classes of a complex vector bundle.

25.1 Orientation on a Vector Bundle

Recall that an orientation on a vector space V of dimension r is an equivalent class of ordered bases, two ordered bases $[u_1 \ \dots \ u_r]$ and $[v_1 \ \dots \ v_r]$ being equivalent if and only if they are related to each other by multiplication by a nonsingular $r \times r$ matrix of positive determinant:

$$[v_1 \ \cdots \ v_r] = [u_1 \ \cdots \ u_r]A, \qquad \det A > 0.$$

Thus, every vector space has exactly two orientations.

Each ordered basis $v_1, \dots, v_r$ determines a nonzero element $v_1 \wedge \cdots \wedge v_r$ in the rth exterior power $\bigwedge^r V$ (Lemma 19.7). If

$$[v_1 \ \cdots \ v_r] = [u_1 \ \cdots \ u_r]A,$$

then by the same calculation as in Proposition 16.8,

$$v_1 \wedge \cdots \wedge v_r = (\det A)\, u_1 \wedge \cdots \wedge u_r.$$

Thus, an orientation on V can also be represented by a nonzero element in $\bigwedge^r V \simeq \mathbb{R}$, with two nonzero elements representing the same orientation if and only if they are positive multiples of each other.

This suggests how to define orientation on a vector bundle.

Definition 25.1. An ***orientation*** on a vector bundle $E \to M$ of rank r is an equivalence class of nowhere-vanishing C^∞ sections of the line bundle $\bigwedge^r E$, two such sections s, s' being equivalent if and only if they are related by multiplication by a positive function on M: $s' = fs$, $f > 0$.

Intuitively this means that each fiber E_p has an orientation which varies with $p \in M$ in a C^∞ way. By Problem 25.1, a line bundle has a nowhere-vanishing C^∞ section if and only if it is a trivial bundle, so we have the following proposition.

Proposition 25.2. *A vector bundle E of rank r has an orientation if and only if the line bundle $\bigwedge^r E$ is trivial.*

If a vector bundle has an orientation, we say that it is ***orientable***. On a connected manifold M an orientable vector bundle E of rank r has exactly two orientations, corresponding to the two connected components of $\bigwedge^r E$. An orientable vector bundle together with a choice of orientation is said to be ***oriented***. In this language, a manifold is orientable if and only if its tangent bundle is an orientable vector bundle.

25.2 Characteristic Classes of an Oriented Vector Bundle

Let $E \to M$ be an oriented Riemannian bundle of rank r, and ∇ a connection on E compatible with the metric. A frame $e = [e_1 \ \cdots \ e_r]$ for E over an open set U is said to be ***positively oriented*** if at each point p in M it agrees with the orientation on E. We agree to use only positively oriented orthonormal frames to compute the connection matrix ω and the curvature matrix Ω of ∇. By Propositions 11.4 and 11.5, relative to the orthonormal frame e, both ω and Ω are skew-symmetric.

If $\bar{e}$ is another positively oriented frame for E over U, then

$$\bar{e} = ea \tag{25.1}$$

for a C^∞ function $a\colon U \to \mathrm{SO}(r)$. Let $\bar{\Omega}$ be the curvature matrix relative to the frame $\bar{e}$. Under the change of frame (25.1) the curvature matrix Ω transforms as in Theorem 22.1(ii):

$$\bar{\Omega} = a^{-1}\Omega a,$$

but now a is a special orthogonal matrix at each point.

Hence, to get a global form $P(\Omega)$ on M, we do not need the polynomial $P(X)$ to be invariant under conjugation by all elements of the general linear group $\mathrm{GL}(r,\mathbb{R})$, but only by elements of the special orthogonal group $\mathrm{SO}(r)$. A polynomial $P(X)$ for $X \in \mathbb{R}^{r\times r}$ is $\mathrm{Ad}\big(\mathrm{SO}(r)\big)$-***invariant*** if

$$P(A^{-1}XA) = P(X) \quad \text{for all } A \in \mathrm{SO}(r).$$

Of course, every $\mathrm{Ad}\big(\mathrm{GL}(r,\mathbb{R})\big)$-invariant polynomial is also $\mathrm{Ad}\big(\mathrm{SO}(r)\big)$-invariant, but might there be an $\mathrm{Ad}\big(\mathrm{SO}(r)\big)$-invariant polynomial that is not $\mathrm{Ad}\big(\mathrm{GL}(r,\mathbb{R})\big)$-invariant? Such a polynomial P would give us a new characteristic class. Note that

for an oriented Riemannian bundle, the curvature matrix Ω relative to an orthonormal frame is skew-symmetric, so the polynomial P need not be defined on all of $\mathfrak{gl}(r,\mathbb{R})$, but only on the subspace of skew-symmetric matrices. The space of all $r\times r$ skew-symmetric matrices is the Lie algebra of $\mathrm{SO}(r)$, denoted by $\mathfrak{so}(r)$. Let $X=[x^i_j]$ be an $r\times r$ skew-symmetric matrix of indeterminates. Thus, $x^j_i=-x^i_j$ for $1\le i<j\le r$, and $x^i_i=0$ for all i. A ***polynomial*** on $\mathfrak{so}(r)$ is an element of $\mathbb{R}[x^i_j]$, where the x^i_j, $1\le i,j\le r$, are indeterminates satisfying the skew-symmetry condition. Alternatively, we may view a polynomial on $\mathfrak{so}(r)$ as an element of the free polynomial ring $\mathbb{R}[x^i_j]$, $1\le i<j\le r$, where the x^i_j are algebraically independent elements over $\mathbb{R}$. We now have a purely algebraic problem:

Problem. What are the $\mathrm{Ad}\big(\mathrm{SO}(r)\big)$-invariant polynomials on $\mathfrak{so}(r)$, the $r\times r$ skew-symmetric matrices?

We already know some of these, namely the $\mathrm{Ad}\big(\mathrm{GL}(r,\mathbb{R})\big)$-invariant polynomials, which are generated as a ring either by the trace polynomials or by the coefficients of the characteristic polynomial. It turns out that for r odd, these are the only $\mathrm{Ad}\big(\mathrm{SO}(r)\big)$-invariant polynomials. For r even, the ring of $\mathrm{Ad}\big(\mathrm{SO}(r)\big)$-invariant polynomials has an additional generator called the ***Pfaffian***, which is a square root of the determinant.

25.3 The Pfaffian of a Skew-Symmetric Matrix

Two $r\times r$ matrices X and Y with entries in a field F are said to be ***congruent*** if there is an $r\times r$ nonsingular matrix $A\in\mathrm{GL}(r,F)$ such that $Y=A^TXA$. By a theorem in linear algebra [13, Th. 8.1, p. 586], every skew-symmetric matrix X over any field F is congruent to a matrix of the form:

$$A^TXA=\begin{bmatrix} S & & & & & \\ & \ddots & & & & \\ & & S & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix},$$



Johann Friedrich Pfaff (1765–1825)

where S is the 2×2 skew-symmetric matrix

$$S=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Since A is nonsingular, it follows that a nonsingular skew-symmetric matrix X has even rank.

If X is nonsingular, then

$$A^T XA = \begin{bmatrix} S & & \\ & \ddots & \\ & & S \end{bmatrix}.$$

Moreover, for such a matrix X,

$$\det(A^T XA) = (\det A)^2(\det X) = (\det S)\cdots(\det S) = 1.$$

So

$$\det X = \left(\frac{1}{\det A}\right)^2 \in F.$$

Thus, the determinant of a skew-symmetric matrix is a perfect square in the field. In fact, more is true.

Theorem 25.3. *Let* $X = [x^i_j]$ *be a* $2m \times 2m$ *skew-symmetric matrix of indeterminates. Then* $\det(X)$ *is a perfect square in the polynomial ring* $\mathbb{Z}[x^i_j]$.

Proof. Let $\mathbb{Q}(x^i_j)$ be the field generated over $\mathbb{Q}$ by the indeterminates x^i_j. We know that $\det X \neq 0$ in $\mathbb{Z}[x^i_j]$, because there exist values of x^i_j in $\mathbb{Z}$ for which $\det X \neq 0$. Thus, $X = [x^i_j]$ is a $2m \times 2m$ nonsingular skew-symmetric matrix over the field $\mathbb{Q}(x^i_j)$. By the discussion above, there is a matrix $A \in \mathrm{GL}\left(2m, \mathbb{Q}(x^i_j)\right)$ such that

$$\det X = \left(\frac{1}{\det A}\right)^2 \in \mathbb{Q}(x^i_j).$$

This proves that $\det X$ is a perfect square in $\mathbb{Q}(x^i_j)$, the fraction field of the integral domain $\mathbb{Z}[x^i_j]$. Suppose

$$\det X = \left(\frac{p(x)}{q(x)}\right)^2,$$

where $p(x), q(x) \in \mathbb{Z}[x^i_j]$ are relatively prime in $\mathbb{Z}[x^i_j]$. Then

$$q(x)^2 \det X = p(x)^2 \in \mathbb{Z}[x^i_j].$$

Since $\mathbb{Z}[x^i_j]$ is a unique factorization domain, $q(x)$ must divide $p(x)$, but $q(x)$ and $p(x)$ are relatively prime, so the only possibilities are $q(x) = \pm 1$. It follows that $\det X = p(x)^2$ in $\mathbb{Z}[x^i_j]$. □

The ***Pfaffian*** of a $2m \times 2m$ skew-symmetric matrix X of indeterminates is defined to be a polynomial $\mathrm{Pf}(X)$ such that

$$\det(X) = (\mathrm{Pf}(X))^2;$$

however, since any perfect square in a ring has two square roots (as long as the characteristic of the ring is not 2), a normalization condition is necessary to determine the sign of $\mathrm{Pf}(X)$. We adopt the convention that on the standard skew-symmetric matrix

$$J_{2m} = \begin{bmatrix} S & & \\ & \ddots & \\ & & S \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

we have $\mathrm{Pf}(J_{2m}) = 1$. The Pfaffian $\mathrm{Pf}(X)$ is a polynomial of degree m on $\mathfrak{so}(2m)$.

Example 25.4. If

$$X = \begin{bmatrix} 0 & x_2^1 \\ -x_2^1 & 0 \end{bmatrix},$$

then $\det(X) = (x_2^1)^2$ and $\mathrm{Pf}(X) = \pm x_2^1$. Since $\mathrm{Pf}(S) = 1$, the sign must be positive. Thus, $\mathrm{Pf}(X) = x_2^1$.

Proposition 25.5. *Let $A = [a_j^i]$ and $X = [x_j^i]$ be $2m \times 2m$ matrices of indeterminates with X skew-symmetric. Then*

$$\mathrm{Pf}(A^T XA) = \det(A)\,\mathrm{Pf}(X) \in \mathbb{Z}[a_j^i, x_j^i].$$

Proof. By the properties of the determinant (Example B.3),

$$\det(A^T XA) = (\det A)^2 \det X. \tag{25.2}$$

Since $A^T XA$ and X are both skew-symmetric, (25.2) is equivalent to

$$\left(\mathrm{Pf}(A^T XA)\right)^2 = (\det A)^2 (\mathrm{Pf} X)^2.$$

Therefore,

$$\mathrm{Pf}(A^T XA) = \pm \det(A)\,\mathrm{Pf}(X). \tag{25.3}$$

Since $\mathrm{Pf}(A^T XA)$ and $\det(A)\,\mathrm{Pf}(X)$ are two uniquely defined elements of the ring $\mathbb{Z}[a_j^i, x_j^i]$, the sign does not depend on A or X. To determine the sign, it suffices to evaluate (25.3) on some particular A and X with integer entries. Taking $A = I_{2m} =$ the $2m \times 2m$ identity matrix and $X = J_{2m}$, we get

$$\mathrm{Pf}(A^T XA) = \mathrm{Pf}(J_{2m}) = 1 = \det(A)\,\mathrm{Pf}(X).$$

Therefore, the sign is $+$ and

$$\mathrm{Pf}(A^T XA) = \det(A)\,\mathrm{Pf}(X). \qquad \square$$

From the proposition, it follows that if $X \in \mathfrak{so}(r)$ and $A \in \mathrm{SO}(r)$, then

$$\mathrm{Pf}(A^{-1} XA) = \mathrm{Pf}(A^T XA) = \det(A)\,\mathrm{Pf}(X) = \mathrm{Pf}(X).$$

Hence, the Pfaffian $\mathrm{Pf}(X)$ is an $\mathrm{Ad}\big(\mathrm{SO}(r)\big)$-invariant polynomial on $r \times r$ skew-symmetric matrices.

25.4 The Euler Class

If $a \in \mathrm{SO}(2m)$, then $a^{-1} = a^T$ and $\det a = 1$. Coming back to the curvature matrix Ω relative to a positively oriented orthonormal frame e of an oriented Riemannian bundle $E \to M$ over an open set U, we see that if $\bar{\Omega}$ is the curvature matrix relative to another positively oriented orthonormal frame $\bar{e}$ for $E|_U$, then by Theorem 22.1(ii) the Pfaffian satisfies

$$\begin{aligned}\mathrm{Pf}(\bar{\Omega}) &= \mathrm{Pf}(a^{-1}\Omega a) = \mathrm{Pf}(a^T\Omega a)\\ &= \det(a)\,\mathrm{Pf}(\Omega) = \mathrm{Pf}(\Omega).\end{aligned}$$

So $\mathrm{Pf}(\Omega)$ is independent of the positively oriented orthonormal frame e. As before, $\mathrm{Pf}(\Omega)$ defines a global form on M. One can prove that this global form is closed and that its cohomology class $[\mathrm{Pf}(\Omega)] \in H^{2m}(M)$ is independent of the connection [17, pp. 310–311]. Instead of doing this, we will give an alternate construction in Section 32 valid for any Lie group. The class $e(E) := [\mathrm{Pf}((1/2\pi)\Omega)]$ is called the ***Euler class*** of the oriented Riemannian bundle E.

25.5 Generalized Gauss–Bonnet Theorem

We can finally state, though not prove, a generalization of the Gauss–Bonnet theorem to higher dimensions.

Theorem 25.6. ([19, Vol. 5, Ch. 13, Th. 26, p.404], [17, p. 311]) *Let M be a compact oriented Riemannian manifold M of dimension $2m$, and ∇ a metric connection on its tangent bundle TM with curvature matrix Ω relative to a positively oriented orthonormal frame. Then*

$$\int_M \mathrm{Pf}\left(\frac{1}{2\pi}\Omega\right) = \chi(M).$$

Example 25.7. When M is a compact oriented surface and e_1, e_2 is a positively oriented orthonormal frame on M, we found the connection and curvature matrices to be

$$\omega = \begin{bmatrix} 0 & \omega_2^1 \\ -\omega_2^1 & 0 \end{bmatrix}, \quad \Omega = d\omega + \omega \wedge \omega = d\omega = \begin{bmatrix} 0 & d\omega_2^1 \\ -d\omega_2^1 & 0 \end{bmatrix}.$$

(see Section 12.1.) By Theorem 12.3, $d\omega_2^1 = K\theta^1 \wedge \theta^2$, where K is the Gaussian curvature and θ^1, θ^2 the dual frame to e_1, e_2. So

$$\mathrm{Pf}(\Omega) = d\omega_2^1 = K\theta^1 \wedge \theta^2.$$

From the generalized Gauss–Bonnet theorem, we get

$$\int_M e(TM) = \int_M \mathrm{Pf}\left(\frac{1}{2\pi}\Omega\right) = \int_M \frac{1}{2\pi} K\theta^1 \wedge \theta^2 = \chi(M),$$

which is the classical Gauss–Bonnet theorem for a compact oriented Riemannian 2-manifold.

25.6 Hermitian Metrics

On the complex vector space $\mathbb{C}^n$, the ***Hermitian inner product*** is given by

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j \bar{w}_j \quad \text{for } z, w \in \mathbb{C}^n.$$

Extracting the properties of the Hermitian inner product, we obtain the definition of a complex inner product on a complex vector space.

Definition 25.8. A ***complex inner product*** on a complex vector space V is a positive-definite, Hermitian-symmetric, sesquilinear form $\langle\ ,\ \rangle\colon V \times V \to \mathbb{C}$. This means that for $u, v, w \in V$ and $a, b \in \mathbb{C}$,

(i) (positive-definiteness) $\langle v, v\rangle \geq 0$; the equality holds if and only if $v = 0$.
(ii) (Hermitian symmetry) $\langle u, v\rangle = \overline{\langle v, u\rangle}$.
(iii) (sesquilinearity) $\langle\ ,\ \rangle$ is linear in the first argument and conjugate linear in the second argument:

$$\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle,$$
$$\langle w, au + bv\rangle = \bar{a}\langle w, u\rangle + \bar{b}\langle w, v\rangle.$$

In fact, for sesquilinearity it is enough to check linearity in the first argument. Conjugate linearity in the second argument follows from linearity in the first argument and Hermitian symmetry.

A ***Hermitian metric*** on a C^∞ complex vector bundle $E \to M$ assigns to each $p \in M$ a complex inner product $\langle\ ,\ \rangle_p$ on the fiber E_p; the assignment is required to be C^∞ in the following sense: if s and t are C^∞ sections of E, then $\langle s, t\rangle$ is a C^∞ complex-valued function on M. A complex vector bundle on which there is given a Hermitian metric is called a ***Hermitian bundle***.

It is easy to check that a finite positive linear combination $\sum a_i\langle\ ,\ \rangle_i$ of complex inner products on a complex vector space V is again a complex inner product. A partition of unity argument as in Theorem 10.8 then proves the existence of a C^∞ Hermitian metric on any C^∞ complex vector bundle.

25.7 Connections and Curvature on a Complex Vector Bundle

A ***connection*** on a complex vector bundle $E \to M$ is defined in exactly the same way as a connection on a real vector bundle (Definition 10.1): it is an $\mathbb{R}$-bilinear map $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ that is $\mathcal{F}$-linear in the first argument, $\mathbb{C}$-linear in the second argument, and satisfies the Leibniz rule in the second argument. Just as in Theorem 10.6, every C^∞ complex vector bundle E over a manifold M has a connection.

On a Hermitian bundle $E \to M$, a connection ∇ is called a ***metric connection*** and is said to be ***compatible with the Hermitian metric*** if for all $X \in \mathfrak{X}(M)$ and $s, t \in \Gamma(E)$,

$$X\langle s, t\rangle = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle.$$

Relative to a frame $e_1,\dots,e_r$ over U for a complex vector bundle E, a connection ∇ can be represented by a matrix $[\omega^i_j]$ of complex-valued 1-forms:

$$\nabla_X e_j = \sum_{i=1}^{r} \omega^i_j(X) e_i.$$

This is just like the real case except that now ω^i_j are *complex-valued* 1-forms on U. For $X,Y \in \mathfrak{X}(M)$, the curvature $R(X,Y)\colon \Gamma(E) \to \Gamma(E)$ of the connection is defined as before:

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}.$$

Relative to a frame $e_1,\dots,e_r$ for E over U, curvature is represented by a matrix $[\Omega^i_j]$ of complex-valued 2-forms. We again have the structural equation

$$\Omega = d\omega + \omega \wedge \omega.$$

25.8 Chern Classes

Under a change of frame $\bar{e} = ea$, where $a \in \mathrm{GL}(r,\mathbb{C})$, the connection and curvature matrices transform by the formulas (Theorem 22.1)

$$\bar{\omega} = a^{-1}\omega a + a^{-1}da,$$

$$\bar{\Omega} = a^{-1}\Omega a.$$

Thus, a complex polynomial Q on $\mathfrak{gl}(r,\mathbb{C})$ that is invariant under conjugation by elements of $\mathrm{GL}(r,\mathbb{C})$ will define a global form $Q(\Omega)$ on M. As before, one shows that $[Q(\Omega)]$ is closed and that the cohomology class $[Q(\Omega)]$ is independent of the connection. Taking $Q(\Omega) = \det(I + \frac{i}{2\pi}\Omega)$, we obtain the ***Chern classes*** $c_i(E)$ of E from

$$\det\left(I + \frac{\sqrt{-1}}{2\pi}\Omega\right) = 1 + c_1(E) + \cdots + c_r(E).$$

The same argument as in Section 24 proves that Chern classes satisfy the naturality property and the Whitney product formula.

Problems

25.1. Trivial line bundle
A ***line bundle*** is a vector bundle of rank 1. Prove that if a C^∞ line bundle $E \to M$ has a nowhere-vanishing C^∞ section, then it is a trivial bundle.

25.2. The determinant and the Pfaffian
For

$$X = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix},$$

find $\det(X)$ and $\mathrm{Pf}(X)$.

25.3. Odd-dimensional skew-symmetric matrices
Using the definition of a skew-symmetric matrix $A^T = -A$, prove that for r odd, an $r \times r$ skew-symmetric matrix is necessarily singular.

§26 Some Applications of Characteristic Classes

The Pontrjagin classes, the Euler class, and the Chern classes are collectively called ***characteristic classes*** of a real or complex vector bundle. They are fundamental diffeomorphism invariants of a vector bundle. Applied to the tangent bundle of a manifold, they yield diffeomorphism invariants of a manifold. We now give a sampling, without proof, of their applications in topology and geometry.

26.1 The Generalized Gauss–Bonnet Theorem

For a compact, oriented Riemannian manifold M of even dimension $2n$, the Euler class $e(TM)$ of the tangent bundle is represented by a closed $2n$-form in the entries of the curvature matrix of M. As we said earlier, the generalized Gauss–Bonnet theorem states that [17, p. 311]

$$\int_M e(TM) = \chi(M).$$

26.2 Characteristic Numbers

On a compact oriented manifold M of dimension $4n$, a monomial $p_1^{i_1} \cdots p_n^{i_n}$ has total degree

$$4i_1 + 8i_2 + \cdots + 4ni_n.$$

If the total degree happens to be $4n$, then the integral $\int_M p_1^{i_1} \cdots p_n^{i_n}$ is defined. It is called a ***Pontrjagin number*** of M. The Pontrjagin numbers are diffeomorphism invariants of the manifold M; in fact, they turn out to be topological invariants as well.

The complex analogue of a smooth manifold is a ***complex manifold***. It is a second countable, Hausdorff topological space that is locally homeomorphic to an open subset of $\mathbb{C}^n$; moreover, if $\phi_\alpha \colon U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{C}^n$ and $\phi_\beta \colon U_\beta \to \phi_\beta(U_\beta) \subset \mathbb{C}^n$ are two such homeomorphisms, then the transition function $\phi_\beta \circ \phi_\alpha^{-1} \colon \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is required to be holomorphic. If $(U, z^1, \ldots, z^n)$ is a chart of the complex manifold, then the holomorphic tangent space T_pM has basis $(\partial/\partial z^1)_p, \ldots, (\partial/\partial z^n)_p$. Just as in the real case, the disjoint union $\coprod_{p \in M} T_pM$ has the structure of a complex vector bundle over M, called the ***holomorphic tangent bundle***. Let $c_1, \ldots, c_n$ be the Chern classes of the holomorphic tangent bundle of a complex manifold of complex dimension n. The degree of the monomial $c_1^{i_1} \cdots c_n^{i_n}$ is $2i_1 + \cdots + 2ni_n$. If this degree happens to be the same as the real dimension $2n$ of the manifold, then the integral $\int_M c_1^{i_1} \cdots c_n^{i_n}$ is defined and is called a ***Chern number*** of M. The Chern numbers are diffeomorphism invariants and also topological invariants of a complex manifold.

26.3 The Cobordism Problem

Two oriented manifolds M_1 and M_2 are said to be ***cobordant*** if there exists an oriented manifold N with boundary such that

$$\partial N = M_1 - M_2,$$

where $-M_2$ denotes M_2 with the opposite orientation. An oriented manifold is cobordant to the empty set if it is the boundary of another oriented manifold. The Pontrjagin numbers give a necessary condition for a compact oriented manifold to be cobordant to the empty set.

Theorem 26.1. *If a compact oriented manifold M of dimension $4n$ is cobordant to the empty set, then all the Pontrjagin numbers of M vanish.*

Proof. Suppose $M = \partial N$ for some manifold N with boundary and that $p_1^{i_1} \cdots p_n^{i_n}$ has degree $4n$. Then

$$\begin{aligned}\int_M p_1^{i_1} \cdots p_n^{i_n} &= \int_{\partial N} p_1^{i_1} \cdots p_n^{i_n} \\ &= \int_N d(p_1^{i_1} \cdots p_n^{i_n}) \quad \text{(by Stokes' theorem)} \\ &= 0,\end{aligned}$$

since Pontrjagin classes are represented by closed forms and a product of closed forms is closed. □

Corollary 26.2. *If two compact oriented manifolds M_1 and M_2 are cobordant, then their respective Pontrjagin numbers are equal.*

Proof. If $M_1 - M_2 = \partial N$, then $\int_{M_1 - M_2} p_1^{i_1} \cdots p_n^{i_n} = 0$ by Theorem 26.1. Hence,

$$\int_{M_1} p_1^{i_1} \cdots p_n^{i_n} = \int_{M_2} p_1^{i_1} \cdots p_n^{i_n}.$$

□

26.4 The Embedding Problem

According to Whitney's embedding theorem, a smooth manifold of dimension n can be embedded in $\mathbb{R}^{2n+1}$. Could one improve on the dimension? In some cases Pontrjagin classes give a necessary condition.

Theorem 26.3. *If a compact oriented manifold M of dimension $4n$ can be embedded as a hypersurface in $\mathbb{R}^{4n+1}$, then all the Pontrjagin classes of M vanish.*

Proof. Since M is compact oriented in $\mathbb{R}^{4n+1}$, it has an outward unit normal vector field and so its normal bundle N in $\mathbb{R}^{4n+1}$ is trivial. From the short exact sequence of C^∞ bundles,

$$0 \to TM \to T\mathbb{R}^{4n+1}|_M \to N \to 0,$$

we obtain a direct sum

$$T\mathbb{R}^{4n+1}|_M \simeq TM \oplus N.$$

By the Whitney product formula for Pontrjagin classes,

$$1 = p(T\mathbb{R}^{4n+1}|_M) = p(TM)p(N) = p(TM).$$

Thus, all the Pontrjagin classes of M are trivial. □

26.5 The Hirzebruch Signature Formula

On a $4n$-dimensional compact oriented manifold M, the intersection of $2n$-dimensional cycles defines a symmetric bilinear pairing

$$\langle\ ,\ \rangle\colon H^{2n}(M;\mathbb{R}) \times H^{2n}(M;\mathbb{R}) \to H^{4n}(M;\mathbb{R}) = \mathbb{R},$$

called the ***intersection form***. The intersection form is represented by a symmetric matrix, which has real eigenvalues. If b^+ is the number of positive eigenvalues and b^- is the number of negative eigenvalues of the symmetric matrix, then the difference $\sigma(M) := b^+ - b^-$ is called the ***signature*** of M. Clearly, it is a topological invariant of the $4n$-dimensional manifold.

In 1953 Hirzebruch found a formula for the signature in terms of Pontrjagin classes.

Theorem 26.4 (Hirzebruch signature formula). *The signature of a compact, smooth, oriented* $4n$*-dimensional manifold is given by*

$$\sigma(M) = \int_M L_n(p_1, \ldots, p_n),$$

where the L_n*'s are the L-polynomials defined in Appendix B.*

26.6 The Riemann–Roch Problem

Many problems in complex analysis and algebraic geometry can be formulated in terms of sections of vector bundles. A ***holomorphic vector bundle*** is a holomorphic map $\pi\colon E \to M$ of complex manifolds that is locally a product $U \times \mathbb{C}^r$ such that the transition functions are holomorphic. If E is a holomorphic vector bundle over a complex manifold M, then the vector space of holomorphic sections of E can be identified with the zeroth cohomology group $H^0(M;E)$ with coefficients in the sheaf of holomorphic sections of E. When M is compact, this vector space turns out to be finite-dimensional, but its dimension is in general not so easy to compute. Instead, the alternating sum

$$\chi(M;E) = \sum_{g=0}^{n} (-1)^q \dim H^q(M;E), \quad n = \dim_{\mathbb{C}} M,$$

is more amenable to computation.

Let L be a complex line bundle over a compact Riemann surface M. The ***degree*** of L is defined to be $\int_M c_1(L)$, where $c_1(L)$ is the first Chern class of L. The classical Riemann–Roch theorem states that for a holomorphic line bundle L over a compact Riemann surface M of genus g,

$$\chi(M;L) = \deg L - g + 1.$$

The HIrzebruch–Riemann–Roch theorem, discovered in 1954, is a generalization to a holomorphic vector bundle E over a compact complex manifold M of complex dimension n

$$\chi(M;E) = \int_M [\mathrm{ch}(E)\,\mathrm{td}(TM)]_n, \tag{26.1}$$

where the Chern character

$$\mathrm{ch}(E) = \mathrm{ch}\big(c_1(E), \dots, c_r(E)\big)$$

and the Todd polynomial

$$\mathrm{td}(TM) = \mathrm{td}\,\big(c_1(M), \dots, c_n(M)\big)$$

are defined as in Appendix B and $[\ ]_n$ means the component of degree n.

In some cases one has vanishing theorems $H^q(M;E) = 0$ for higher cohomology $q \geq 1$. When this happens, $\dim H^0(M;E) = \chi(M;E)$, which can be computed from the Hirzebruch–Riemann–Roch theorem.

Chapter 6

Principal Bundles and Characteristic Classes

A principal bundle is a locally trivial family of groups. It turns out that the theory of connections on a vector bundle can be subsumed under the theory of connections on a principal bundle. The latter, moreover, has the advantage that its connection forms are basis-free.

In this chapter we will first give several equivalent constructions of a connection on a principal bundle, and then generalize the notion curvature to a principal bundle, paving the way to a generalization of characteristic classes to principal bundles. Along the way, we also generalize covariant derivatives to principal bundles.

§27 Principal Bundles

We saw in Section 11 that a connection ∇ on a vector bundle E over a manifold M can be represented by a matrix of 1-forms over a framed open set. For any frame $e = [e_1 \ \cdots \ e_r]$ for E over an open set U, the connection matrix ω_e relative to e is defined by

$$\nabla_X e_j = \sum_i (\omega_e)^i_j(X) e_i$$

for all C^∞ vector fields X over U. If $\bar{e} = [\bar{e}_1 \ \cdots \ \bar{e}_r] = ea$ is another frame for E over U, where $a\colon U \to \mathrm{GL}(r,\mathbb{R})$ is a matrix of C^∞ transition functions, then by Theorem 22.1 the connection matrix ω_e transforms according to the rule

$$\omega_{\bar{e}} = a^{-1}\omega_e a + a^{-1}\,da.$$

Associated to a vector bundle is an object called its ***frame bundle*** $\pi\colon \mathrm{Fr}(E) \to M$; the total space $\mathrm{Fr}(E)$ of the frame bundle is the set of all ordered bases in the fibers of the vector bundle $E \to M$, with a suitable topology and manifold structure. A ***section*** of the frame bundle $\pi\colon \mathrm{Fr}(E) \to M$ over an open set $U \subset M$ is a map $s\colon U \to \mathrm{Fr}(E)$

L.W. Tu, *Differential Geometry*, Graduate Texts in Mathematics 275,
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such that $\pi \circ s = \mathbb{1}_U$, the identity map on U. From this point of view a frame $e = [e_1 \ \cdots \ e_r]$ over U is simply a section of the frame bundle $\mathrm{Fr}(E)$ over U.

Suppose ∇ is a connection on the vector bundle $E \to M$. Miraculously, there exists a matrix-valued 1-form ω on the frame bundle $\mathrm{Fr}(E)$ such that for every frame e over an open set $U \subset M$, the connection matrix ω_e of ∇ is the pullback of ω by the section $e\colon U \to \mathrm{Fr}(E)$ (Theorem 29.10). This matrix-valued 1-form, called an ***Ehresmann connection*** on the frame bundle $\mathrm{Fr}(E)$, is determined uniquely by the connection on the vector bundle E and vice versa. It is an intrinsic object of which a connection matrix ω_e is but a local manifestation. The frame bundle of a vector bundle is an example of a principal G-bundle for the group $G = \mathrm{GL}(r, \mathbb{R})$. The Ehresmann connection on the frame bundle generalizes to a connection on an arbitrary principal bundle.

This section collects together some general facts about principal bundles.



Charles Ehresmann

(1905–1979)

27.1 Principal Bundles

Let E, M, and F be manifolds. We will denote an open cover $\mathfrak{U}$ of M either as $\{U_\alpha\}$ or more simply as an unindexed set $\{U\}$ whose general element is denoted by U. A ***local trivialization*** with fiber F for a smooth surjection $\pi\colon E \to M$ is an open cover $\mathfrak{U} = \{U\}$ for M together with a collection $\{\phi_U\colon \pi^{-1}(U) \to U \times F \mid U \in \mathfrak{U}\}$ of fiber-preserving diffeomorphisms $\phi_U\colon \pi^{-1}(U) \to U \times F$:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi_U} & U \times F \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \eta} & \\ & U. & \end{array}$$

where η is projection to the first factor. A ***fiber bundle*** with fiber F is a smooth surjection $\pi\colon E \to M$ having a local trivialization with fiber F. We also say that it is ***locally trivial*** with fiber F. The manifold E is the ***total space*** and the manifold M the ***base space*** of the fiber bundle.

The ***fiber*** of a fiber bundle $\pi\colon E \to M$ over $x \in M$ is the set $E_x := \pi^{-1}(x)$. Because π is a submersion, by the regular level set theorem ([21], Th. 9.13, p. 96) each fiber E_x is a regular submanifold of E. For $x \in U$, define $\phi_{U,x} := \phi_U|_{E_x}\colon E_x \to \{x\} \times F$ to be the restriction of the trivialization $\phi_U\colon \pi^{-1}(U) \to U \times F$ to the fiber E_x.

Proposition 27.1. *Let $\pi\colon E \to M$ be a fiber bundle with fiber F. If $\phi_U\colon \pi^{-1}(U) \to U \times F$ is a trivialization, then $\phi_{U,x}\colon E_x \to \{x\} \times F$ is a diffeomorphism.*

Proof. The map $\phi_{U,x}$ is smooth because it is the restriction of the smooth map ϕ_U to a regular submanifold. It is bijective because ϕ_U is bijective and fiber-preserving. Its inverse $\phi_{U,x}^{-1}$ is the restriction of the smooth map $\phi_U^{-1}\colon U \times F \to \pi^{-1}(U)$ to the fiber $\{x\} \times F$ and is therefore also smooth. □

A smooth ***right action*** of a Lie group G on a manifold M is a smooth map

$$\mu\colon M \times G \to M,$$

denoted by $x \cdot g := \mu(x,g)$, such that for all $x \in M$ and $g, h \in G$,

(i) $x \cdot e = x$, where e is the identity element of G,
(ii) $(x \cdot g) \cdot h = x \cdot (gh)$.

We often omit the dot and write more simply xg for $x \cdot g$. If there is such a map μ, we also say that G ***acts smoothly on*** M ***on the right***. A ***left action*** is defined similarly. The ***stabilizer*** of a point $x \in M$ under an action of G is the subgroup

$$\mathrm{Stab}(x) := \{g \in G \mid x \cdot g = x\}.$$

The ***orbit*** of $x \in M$ is the set

$$\mathrm{Orbit}(x) := xG := \{x \cdot g \in M \mid g \in G\}.$$

Denote by $\mathrm{Stab}(x)\backslash G$ the set of right cosets of $\mathrm{Stab}(x)$ in G. By the orbit-stabilizer theorem, for each $x \in M$ the map: $G \to \mathrm{Orbit}(x)$, $g \mapsto x \cdot g$ induces a bijection of sets:

$$\begin{aligned} \mathrm{Stab}(x)\backslash G &\longleftrightarrow \mathrm{Orbit}(x), \\ \mathrm{Stab}(x)g &\longleftrightarrow x \cdot g. \end{aligned}$$

The action of G on M is ***free*** if the stabilizer of every point $x \in M$ is the trivial subgroup $\{e\}$.

A manifold M together with a right action of a Lie group G on M is called a ***right*** ***G-manifold*** or simply a ***G-manifold***. A map $f\colon N \to M$ between right G-manifolds is ***right G-equivariant*** if

$$f(x \cdot g) = f(x) \cdot g$$

for all $(x,g) \in N \times G$. Similarly, a map $f\colon N \to M$ between left G-manifolds is ***left G-equivariant*** if

$$f(g \cdot x) = g \cdot f(x)$$

for all $(g,x) \in G \times N$.

A left action can be turned into a right action and vice versa; for example, if G acts on M on the left, then

$$x \cdot g = g^{-1} \cdot x$$

is a right action of G on M. Thus, if N is a right G-manifold and M is a left G-manifold, we say a map $f\colon N \to M$ is ***G-equivariant*** if

$$f(x \cdot g) = f(x) \cdot g = g^{-1} \cdot f(x) \tag{27.1}$$

for all $(x,g) \in N \times G$.

A smooth fiber bundle $\pi\colon P \to M$ with fiber G is a smooth ***principal G-bundle*** if G acts smoothly and freely on P on the right and the fiber-preserving local trivializations

$$\phi_U\colon \pi^{-1}(U) \to U \times G$$

are G-equivariant, where G acts on $U \times G$ on the right by

$$(x,h) \cdot g = (x, hg).$$

Example 27.2 (*Product G-bundles*). The simplest example of a principal G-bundle over a manifold M is the product G-bundle $\eta\colon M \times G \to M$. A trivialization is the identity map on $M \times G$.

Example 27.3 (*Homogenous manifolds*). If G is a Lie group and H is a closed subgroup, then the quotient G/H can be given the structure of a manifold such that the projection map $\pi\colon G \to G/H$ is a principal H-bundle. This is proven in [22, Th. 3.58, p. 120].

Example 27.4 (*Hopf bundle*). The group S^1 of unit complex numbers acts on the complex vector space $\mathbb{C}^{n+1}$ by left multiplication. This action induces an action of S^1 on the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$. The complex projective space $\mathbb{C}P^n$ may be defined as the orbit space of S^{2n+1} by S^1. The natural projection $S^{2n+1} \to \mathbb{C}P^n$ with fiber S^1 turn out to be a principal S^1-bundle. When $n = 1$, $S^3 \to \mathbb{C}P^1$ with fiber S^1 is called the ***Hopf bundle***.

Definition 27.5. Let $\pi_Q\colon Q \to N$ and $\pi_P\colon P \to M$ be principal G-bundles. A ***morphism*** of principal G-bundles is a pair of maps $(\bar{f}\colon Q \to P, f\colon N \to M)$ such that $\bar{f}\colon Q \to P$ is G-equivariant and the diagram

$$\begin{array}{ccc} Q & \xrightarrow{\bar{f}} & P \\ {\scriptstyle \pi_Q}\downarrow & & \downarrow{\scriptstyle \pi_P} \\ N & \xrightarrow[f]{} & M \end{array}$$

commutes.

Proposition 27.6. *If $\pi\colon P \to M$ is a principal G-bundle, then the group G acts transitively on each fiber.*

Proof. Since G acts transitively on $\{x\} \times G$ and the fiber diffeomorphism $\phi_{U,x}\colon P_x \to \{x\} \times G$ is G-equivariant, G must also act transitively on the fiber P_x. □

Lemma 27.7. *For any group G, a right G-equivariant map $f\colon G \to G$ is necessarily a left translation.*

Proof. Suppose that for all $x, g \in G$,

$$f(xg) = f(x)g.$$

Setting $x = e$, the identity element of G, we obtain

$$f(g) = f(e)g = \ell_{f(e)}(g),$$

where $\ell_{f(e)}\colon G \to G$ is left translation by $f(e)$. □

Suppose $\{U_\alpha\}_{\alpha\in A}$ is a local trivialization for a principal G-bundle $\pi\colon P \to M$. Whenever the intersection $U_{\alpha\beta} := U_\alpha \cap U_\beta$ is nonempty, there are two trivializations on $\pi^{-1}(U_{\alpha\beta})$:

$$U_{\alpha\beta} \times G \xleftarrow{\phi_\alpha} \pi^{-1}(U_{\alpha\beta}) \xrightarrow{\phi_\beta} U_{\alpha\beta} \times G.$$

Then $\phi_\alpha \circ \phi_\beta^{-1}\colon U_{\alpha\beta} \times G \to U_{\alpha\beta} \times G$ is a fiber-preserving right G-equivariant map. By Lemma 27.7, it is a left translation on each fiber. Thus,

$$(\phi_\alpha \circ \phi_\beta^{-1})(x,h) = (x, g_{\alpha\beta}(x)h), \tag{27.2}$$

where $(x,h) \in U_{\alpha\beta} \times G$ and $g_{\alpha\beta}(x) \in G$. Because $\phi_\alpha \circ \phi_\beta^{-1}$ is a C^∞ function of x and h, setting $h = e$, we see that $g_{\alpha\beta}(x)$ is a C^∞ function of x. The C^∞ functions $g_{\alpha\beta}\colon U_{\alpha\beta} \to G$ are called ***transition functions*** of the principal bundle $\pi\colon P \to M$ relative to the trivializing open cover $\{U_\alpha\}_{\alpha\in A}$. They satisfy the ***cocycle condition***: for all $\alpha, \beta, \gamma \in A$,

$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma} \quad \text{if } U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing.$$

From the cocycle condition, one can deduce other properties of the transition functions.

Proposition 27.8. *The transition functions $g_{\alpha\beta}$ of a principal bundle $\pi\colon P \to M$ relative to a trivializing open cover $\{U_\alpha\}_{\alpha\in A}$ satisfy the following properties: for all $\alpha, \beta \in A$,*

(i) $g_{\alpha\alpha} =$ *the constant map e,*
(ii) $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ *if $U_\alpha \cap U_\beta \neq \varnothing$.*

Proof. (i) If $\alpha = \beta = \gamma$, the cocycle condition gives

$$g_{\alpha\alpha} g_{\alpha\alpha} = g_{\alpha\alpha}.$$

Hence, $g_{\alpha\alpha} =$ the constant map e.
(ii) if $\gamma = \alpha$, the cocycle condition gives

$$g_{\alpha\beta} g_{\beta\alpha} = g_{\alpha\alpha} = e$$

or

$$g_{\alpha\beta} = g_{\beta\alpha}^{-1} \quad \text{for } U_\alpha \cap U_\beta \neq \varnothing.$$ □

In a principal G-bundle $P \to M$, the group G acts on the right on the total space P, but the transition functions $g_{\alpha\beta}$ in (27.2) are given by left translations by $g_{\alpha\beta}(x) \in G$. This phenomenon is a consequence of Lemma 27.7.

27.2 The Frame Bundle of a Vector Bundle

For any real vector space V, let $\mathrm{Fr}(V)$ be the set of all ordered bases in V. Suppose V has dimension r. We will represent an ordered basis $v_1,\dots,v_r$ by a row vector $v = [v_1 \ \cdots \ v_r]$, so that the general linear group $\mathrm{GL}(r,\mathbb{R})$ acts on $\mathrm{Fr}(V)$ on the right by matrix multiplication

$$\begin{aligned} v\cdot a &= [v_1 \ \cdots \ v_r][a^i_j] \\ &= \left[\sum v_i a^i_1 \ \cdots \ \sum v_i a^i_r\right]. \end{aligned}$$

Fix a point $v \in \mathrm{Fr}(V)$. Since the action of $\mathrm{GL}(r,\mathbb{R})$ on $\mathrm{Fr}(V)$ is clearly transitive and free, i.e., $\mathrm{Orbit}(v) = \mathrm{Fr}(V)$ and $\mathrm{Stab}(v) = \{I\}$, by the orbit-stabilizer theorem there is a bijection

$$\begin{aligned} \phi_v \colon \ \mathrm{GL}(r,\mathbb{R}) = \frac{\mathrm{GL}(r,\mathbb{R})}{\mathrm{Stab}(v)} &\longleftrightarrow \mathrm{Orbit}(v) = \mathrm{Fr}(V), \\ g &\longleftrightarrow vg. \end{aligned}$$

Using the bijection ϕ_v, we put a manifold structure on $\mathrm{Fr}(V)$ in such a way that ϕ_v becomes a diffeomorphism.

If v' is another element of $\mathrm{Fr}(V)$, then $v' = va$ for some $a \in \mathrm{GL}(r,\mathbb{R})$ and

$$\phi_{va}(g) = vag = \phi_v(ag) = (\phi_v \circ \ell_a)(g).$$

Since left multiplication $\ell_a \colon \ \mathrm{GL}(r,\mathbb{R}) \to \mathrm{GL}(r,\mathbb{R})$ is a diffeomorphism, the manifold structure on $\mathrm{Fr}(V)$ defined by ϕ_v is the same as the one defined by ϕ_{va}. We call $\mathrm{Fr}(V)$ with this manifold structure the ***frame manifold*** of the vector space V.

Remark 27.9. A linear isomorphism $\phi \colon V \to W$ induces a C^∞ diffeomorphism $\widetilde{\phi}\colon$ $\mathrm{Fr}(V) \to \mathrm{Fr}(W)$ by

$$\widetilde{\phi}[v_1 \ \cdots \ v_r] = [\phi(v_1) \ \cdots \ \phi(v_r)].$$

Define an action of $\mathrm{GL}(r,\mathbb{R})$ on $\mathrm{Fr}(\mathbb{R}^r)$ by

$$g\cdot[v_1 \ \cdots \ v_r] = [gv_1 \ \cdots \ gv_r].$$

Thus, if $\phi \colon \mathbb{R}^r \to \mathbb{R}^r$ is given by left multiplication by $g \in \mathrm{GL}(r,\mathbb{R})$, then so is the induced map $\widetilde{\phi}$ on the frame manifold $\mathrm{Fr}(\mathbb{R}^r)$.

Example 27.10 (*The frame bundle*). Let $\eta \colon E \to M$ be a C^∞ vector bundle of rank r. We associate to the vector bundle E a C^∞ principal $\mathrm{GL}(r,\mathbb{R})$-bundle $\pi \colon \ \mathrm{Fr}(E) \to M$ as follows. As a set the total space $\mathrm{Fr}(E)$ is defined to be the disjoint union

$$\mathrm{Fr}(E) = \coprod_{x\in M} \mathrm{Fr}(E_x).$$

There is a natural projection map $\pi \colon \ \mathrm{Fr}(E) \to M$ that maps $\mathrm{Fr}(E_x)$ to $\{x\}$.

A local trivialization $\phi_\alpha \colon E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^r$ induces a bijection

$$\widetilde{\phi_\alpha} \colon \operatorname{Fr}(E)|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \operatorname{Fr}(\mathbb{R}^r),$$
$$[v_1 \ \cdots \ v_r] \in \operatorname{Fr}(E_x) \mapsto (x, [\phi_{\alpha,x}(v_1) \ \cdots \ \phi_{\alpha,x}(v_r)]).$$

Via $\widetilde{\phi_\alpha}$ one transfers the topology and manifold structure from $U_\alpha \times \operatorname{Fr}(\mathbb{R}^r)$ to $\operatorname{Fr}(E)|_{U_\alpha}$. This gives $\operatorname{Fr}(E)$ a topology and a manifold structure such that $\pi \colon \operatorname{Fr}(E) \to M$ is locally trivial with fiber $\operatorname{Fr}(\mathbb{R}^r)$. As the frame manifold $\operatorname{Fr}(\mathbb{R}^r)$ is diffeomorphic to the general linear group $\operatorname{GL}(r,\mathbb{R})$, it is easy to check that $\operatorname{Fr}(E) \to M$ is a C^∞ principal $\operatorname{GL}(r,\mathbb{R})$-bundle. We call it the ***frame bundle*** of the vector bundle E.

On a nonempty overlap $U_{\alpha\beta} := U_\alpha \cap U_\beta$, the transition function for the vector bundle E is the C^∞ function $g_{\alpha\beta} \colon U_{\alpha\beta} \to \operatorname{GL}(r,\mathbb{R})$ given by

$$\phi_\alpha \circ \phi_\beta^{-1} \colon U_{\alpha\beta} \times \mathbb{R}^r \to U_{\alpha\beta} \times \mathbb{R}^r,$$
$$(\phi_\alpha \circ \phi_\beta^{-1})(x,w) = \big(x, g_{\alpha\beta}(x)w\big).$$

Since the local trivialization for the frame bundle $\operatorname{Fr}(E)$ is induced from the trivialization $\{U_\alpha, \phi_\alpha\}$ for E, the transition functions for $\operatorname{Fr}(E)$ are induced from the transition functions $\{g_{\alpha\beta}\}$ for E. By Remark 27.9 the transition functions for the open cover $\{\operatorname{Fr}(E)|_{U_\alpha}\}$ of $\operatorname{Fr}(E)$ are the same as the transition functions $g_{\alpha\beta} \colon U_{\alpha\beta} \to \operatorname{GL}(r,\mathbb{R})$ for the vector bundle E, but now of course $\operatorname{GL}(r,\mathbb{R})$ acts on $\operatorname{Fr}(\mathbb{R}^r)$ instead of on $\mathbb{R}^r$.

27.3 Fundamental Vector Fields of a Right Action

Suppose G is a Lie group with Lie algebra $\mathfrak{g}$ and G acts smoothly on a manifold P on the right. To every element $A \in \mathfrak{g}$ one can associate a vector field $\underline{A}$ on P called the ***fundamental vector field on*** P ***associated to*** A: for p in P, define

$$\underline{A}_p = \frac{d}{dt}\bigg|_{t=0} p \cdot e^{tA} \in T_pP.$$

To understand this equation, first fix a point $p \in P$. Then $c_p \colon t \mapsto p \cdot e^{tA}$ is a curve in P with initial point p. By definition, the vector $\underline{A}_p$ is the initial vector of this curve. Thus,

$$\underline{A}_p = c_p'(0) = c_{p*}\left(\frac{d}{dt}\bigg|_{t=0}\right) \in T_pP.$$

As a tangent vector at p is a derivation on germs of C^∞ functions at p, in terms of a C^∞ function f at p,

$$\underline{A}_p f = c_{p*}\left(\frac{d}{dt}\bigg|_{t=0}\right) f = \frac{d}{dt}\bigg|_{t=0} f \circ c_p = \frac{d}{dt}\bigg|_{t=0} f(p \cdot e^{tA}).$$

Proposition 27.11. *For each $A \in \mathfrak{g}$, the fundamental vector field $\underline{A}$ is C^∞ on P.*

Proof. It suffices to show that for every C^∞ function f on P, the function $\underline{A}f$ is also C^∞ on P. Let $\mu\colon P\times G\to P$ be the C^∞ map defining the right action of G on P. For any p in P,

$$\underline{A}_p f = \frac{d}{dt}\bigg|_{t=0} f(p\cdot e^{tA}) = \frac{d}{dt}\bigg|_{t=0} (f\circ\mu)(p,e^{tA}).$$

Since e^{tA} is a C^∞ function of t, and f and μ are both C^∞, the derivative

$$\frac{d}{dt}(f\circ\mu)(p,e^{tA})$$

is C^∞ in p and in t. Therefore, $\underline{A}_p f$ is a C^∞ function of p. □

Recall that $\mathfrak{X}(P)$ denotes the Lie algebra of C^∞ vector fields on the manifold P. The fundamental vector field construction gives rise to a map

$$\sigma\colon \mathfrak{g}\to\mathfrak{X}(P),\quad \sigma(A):=\underline{A}.$$

For p in P, define $j_p\colon G\to P$ by $j_p(g)=p\cdot g$. Computing the differential j_{p*} using the curve $c(t)=e^{tA}$, we obtain the expression

$$j_{p*}(A) = \frac{d}{dt}\bigg|_{t=0} j_p(e^{tA}) = \frac{d}{dt}\bigg|_{t=0} p\cdot e^{tA} = \underline{A}_p. \tag{27.3}$$

This alternate description of fundamental vector fields, $\underline{A}_p = j_{p*}(A)$, shows that the map $\sigma\colon\mathfrak{g}\to\mathfrak{X}(P)$ is linear over $\mathbb{R}$. In fact, σ is a Lie algebra homomorphism (Problem 27.1).

Example 27.12. Consider the action of a Lie group G on itself by right multiplication. For $p\in G$, the map $j_p\colon G\to G$, $j_p(g)=p\cdot g=\ell_p(g)$ is simply left multiplication by p. By (27.3), for $A\in\mathfrak{g}$, $\underline{A}_p=\ell_{p*}(A)$. Thus, for the action of G on G by right multiplication, the fundamental vector field $\underline{A}$ on G is precisely the left-invariant vector field generated by A. In this sense the fundamental vector field of a right action is a generalization of a left-invariant vector field on a Lie group.

For g in a Lie group G, let $c_g\colon G\to G$ be conjugation by g: $c_g(x)=gxg^{-1}$. The ***adjoint representation*** is defined to be the differential of the conjugation map: $\mathrm{Ad}(g)=(c_g)_*\colon\mathfrak{g}\to\mathfrak{g}$.

Proposition 27.13. *Suppose a Lie group G acts smoothly on a manifold P on the right. Let $r_g\colon P\to P$ be the right translation $r_g(p)=p\cdot g$. For $A\in\mathfrak{g}$ the fundamental vector field $\underline{A}$ on P satisfies the following equivariance property:*

$$r_{g*}\underline{A} = \underline{(\mathrm{Ad}\,g^{-1})A}.$$

Proof. We need to show that for every p in P, $r_{g*}(\underline{A}_p)=\underline{(\mathrm{Ad}\,g^{-1})A}_{pg}$. For x in G,

$$(r_g\circ j_p)(x)=pxg=pgg^{-1}xg=j_{pg}(g^{-1}xg)=(j_{pg}\circ c_{g^{-1}})(x).$$

By the chain rule,

$$r_{g*}(\underline{A}_p)=r_{g*}j_{p*}(A)=j_{pg*}(c_{g^{-1}})_*(A)=j_{pg*}\big((\mathrm{Ad}\,g^{-1})A\big)=\underline{(\mathrm{Ad}\,g^{-1})A}_{pg}.$$ □

27.4 Integral Curves of a Fundamental Vector Field

In this section suppose a Lie group G with Lie algebra $\mathfrak{g} := \operatorname{Lie}(G)$ acts smoothly on the right on a manifold P.

Proposition 27.14. *For $p \in P$ and $A \in \mathfrak{g}$, the curve $c_p(t) = p \cdot e^{tA}$, $t \in \mathbb{R}$, is the integral curve of the fundamental vector field $\underline{A}$ through p.*

Proof. We need to show that $c_p'(t) = \underline{A}_{c_p(t)}$ for all $t \in \mathbb{R}$ and all $p \in P$. It is essentially a sequence of definitions:

$$c_p'(t) = \left.\frac{d}{ds}\right|_{s=0} c_p(t+s) = \left.\frac{d}{ds}\right|_{s=0} pe^{tA}e^{sA} = \underline{A}_{pe^{tA}} = \underline{A}_{c_p(t)}. \qquad \square$$

Proposition 27.15. *The fundamental vector field $\underline{A}$ on a manifold P vanishes at a point p in P if and only if A is in the Lie algebra of* $\operatorname{Stab}(p)$.

Proof. ($\Leftarrow$) If $A \in \operatorname{Lie}\big(\operatorname{Stab}(p)\big)$, then $e^{tA} \in \operatorname{Stab}(p)$, so

$$\underline{A}_p = \left.\frac{d}{dt}\right|_{t=0} p \cdot e^{tA} = \left.\frac{d}{dt}\right|_{t=0} p = 0.$$

($\Rightarrow$) Suppose $\underline{A}_p = 0$. Then the constant map $\gamma(t) = p$ is an integral curve of $\underline{A}$ through p, since

$$\gamma'(t) = 0 = \underline{A}_p = \underline{A}_{\gamma(t)}.$$

On the other hand, by Proposition 27.14, $c_p(t) = p \cdot e^{tA}$ is also an integral curve of $\underline{A}$ through p. By the uniqueness of the integral curve through a point, $c_p(t) = \gamma(t)$ or $p \cdot e^{tA} = p$ for all $t \in \mathbb{R}$. This implies that $e^{tA} \in \operatorname{Stab}(p)$ and therefore $A \in \operatorname{Lie}\big(\operatorname{Stab}(p)\big)$. $\square$

Corollary 27.16. *For a right action of a Lie group G on a manifold P, let $p \in P$ and $j_p \colon G \to P$ be the map $j_p(g) = p \cdot g$. Then the kernel* $\ker j_{p*}$ *of the differential of j_p at the identity*

$$j_{p*} = (j_p)_{*,e} \colon \mathfrak{g} \to T_pP$$

is $\operatorname{Lie}\big(\operatorname{Stab}(p)\big)$.

Proof. For $A \in \mathfrak{g}$, we have $\underline{A}_p = j_{p*}(A)$ by (27.3). Thus,

$$\begin{aligned} A \in \ker j_{p*} \quad &\Longleftrightarrow \quad j_{p*}(A) = 0 \\ &\Longleftrightarrow \quad \underline{A}_p = 0 \\ &\Longleftrightarrow \quad A \in \operatorname{Lie}\big(\operatorname{Stab}(p)\big) \quad \text{(by Proposition 27.15).} \end{aligned} \qquad \square$$

27.5 Vertical Subbundle of the Tangent Bundle TP

Throughout this section, G is a Lie group with Lie algebra $\mathfrak{g}$ and $\pi\colon P \to M$ is a principal G-bundle. On the total space P there is a natural notion of vertical tangent vectors. We will show that the vertical tangent vectors on P form a trivial subbundle of the tangent bundle TP.

By the local triviality of a principal bundle, at every point $p \in P$ the differential $\pi_{*,p}\colon T_pP \to T_{\pi(p)}M$ of the projection π is surjective. The ***vertical tangent subspace*** $\mathcal{V}_p \subset T_pP$ is defined to be $\ker \pi_{*,p}$. Hence, there is a short exact sequence of vector spaces

$$0 \to \mathcal{V}_p \longrightarrow T_pP \xrightarrow{\pi_{*,p}} T_{\pi(p)}M \to 0, \tag{27.4}$$

and

$$\dim \mathcal{V}_p = \dim T_pP - \dim T_{\pi(p)}M = \dim G.$$

An element of $\mathcal{V}_p$ is called a ***vertical tangent vector*** at p.

Proposition 27.17. *For any $A \in \mathfrak{g}$, the fundamental vector field $\underline{A}$ is vertical at every point $p \in P$.*

Proof. With $j_p\colon G \to P$ defined as usual by $j_p(g) = p \cdot g$,

$$(\pi \circ j_p)(g) = \pi(p \cdot g) = \pi(p).$$

Since $\underline{A}_p = j_{p*}(A)$ by (27.3), and $\pi \circ j_p$ is a constant map,

$$\pi_{*,p}(\underline{A}_p) = (\pi_{*,p} \circ j_{p*})(A) = (\pi \circ j_p)_*(A) = 0. \qquad \square$$

Thus, in case P is a principal G-bundle, we can refine Corollary 27.16 to show that j_{p*} maps $\mathfrak{g}$ into the vertical tangent space:

$$(j_p)_{*,e}\colon \mathfrak{g} \to \mathcal{V}_p \subset T_pP.$$

In fact, this is an isomorphism.

Proposition 27.18. *For $p \in P$, the differential at e of the map $j_p\colon G \to P$ is an isomorphism of $\mathfrak{g}$ onto the vertical tangent space: $j_{p*} = (j_p)_{*,e}\colon \mathfrak{g} \xrightarrow{\sim} \mathcal{V}_p$.*

Proof. By Corollary 27.16, $\ker j_{p*} = \operatorname{Lie}\big(\operatorname{Stab}(p)\big)$. Since G acts freely on P, the stabilizer of any point $p \in P$ is the trivial subgroup $\{e\}$. Thus, $\ker j_{p*} = 0$ and j_{p*} is injective. By Proposition 27.17, the image j_{p*} lies in the vertical tangent space $\mathcal{V}_p$. Since $\mathfrak{g}$ and $\mathcal{V}_p$ have the same dimension, the injective linear map $j_{p*}\colon \mathfrak{g} \to \mathcal{V}_p$ has to be an isomorphism. $\square$

Corollary 27.19. *The vertical tangent vectors at a point of a principal bundle are precisely the fundamental vectors.*

Let $B_1, \ldots, B_\ell$ be a basis for the Lie algebra $\mathfrak{g}$. By the proposition, the fundamental vector fields $\underline{B_1}, \ldots, \underline{B_\ell}$ on P form a basis of $\mathcal{V}_p$ at every point $p \in P$. Hence, they span a trivial subbundle $\mathcal{V} := \coprod_{p \in P} \mathcal{V}_p$ of the tangent bundle TP. We call $\mathcal{V}$ the ***vertical subbundle*** of TP.

As we learned in Section 20.5, the differential $\pi_*\colon TP \to TM$ of a C^∞ map $\pi\colon P \to M$ induces a bundle map $\tilde{\pi}_*\colon TP \to \pi^*TM$ over P, given by

$$\begin{array}{ccc} TP & \xrightarrow{\tilde{\pi}_*} & \pi^*TM \\ & \searrow \qquad \swarrow & \\ & P & \end{array} \qquad T_pP \ni X_p \mapsto (p, \pi_{*,p}X_p).$$

The map $\tilde{\pi}_*$ is surjective because it sends the fiber T_pP onto the fiber $(\pi^*TM)_p \simeq T_{\pi(p)}M$. Its kernel is precisely the vertical subbundle $\mathcal{V}$ by (27.4). Hence, $\mathcal{V}$ fits into a short exact sequence of vector bundles over P:

$$0 \to \mathcal{V} \longrightarrow TP \xrightarrow{\tilde{\pi}_*} \pi^*TM \to 0. \tag{27.5}$$

27.6 Horizontal Distributions on a Principal Bundle

On the total space P of a smooth principal bundle $\pi\colon P \to M$, there is a well-defined vertical subbundle $\mathcal{V}$ of the tangent bundle TP. We call a subbundle $\mathcal{H}$ of TP a ***horizontal distribution*** on P if $TP = \mathcal{V} \oplus \mathcal{H}$ as vector bundles; in other words, $T_pP = \mathcal{V}_p + \mathcal{H}_p$ and $\mathcal{V}_p \cap \mathcal{H}_p = 0$ for every $p \in P$. In general, there is no canonically defined horizontal distribution on a principal bundle.

A ***splitting*** of a short exact sequence of vector bundles $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ over a manifold P is a bundle map $k\colon C \to B$ such that $j \circ k = \mathbb{1}_C$, the identity bundle map on C.

Proposition 27.20. *Let*

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0 \tag{27.6}$$

be a short exact sequence of vector bundles over a manifold P. Then there is a one-to-one correspondence

$$\{\textit{subbundles } H \subset B \mid B = i(A) \oplus H\} \longleftrightarrow \{\textit{splittings } k\colon C \to B \textit{ of } (27.6)\}.$$

Proof. If H is a subbundle of B such that $B = i(A) \oplus H$, then there are bundle isomorphisms $H \simeq B/i(A) \simeq C$. Hence, C maps isomorphically onto H in B. This gives a splitting $k\colon C \to B$.

If $k\colon C \to B$ is a splitting, let $H := k(C)$, which is a subbundle of B. Moreover, if $i(a) = k(c)$ for some $a \in A$ and $c \in C$, then

$$0 = ji(a) = jk(c) = c.$$

Hence, $i(A) \cap k(C) = 0$.

Finally, to show that $B = i(A) + k(C)$, let $b \in B$. Then

$$j\big(b - kj(b)\big) = j(b) - j(b) = 0.$$

By the exactness of (27.6), $b - kj(b) = i(a)$ for some $a \in A$. Thus,

$$b = i(a) + kj(b) \in i(A) + k(C).$$

This proves that $B = i(A) + k(C)$ and therefore $B = i(A) \oplus k(C)$. □

As we just saw in the preceding section, for every principal bundle $\pi\colon P \to M$ the vertical subbundle $\mathcal{V}$ fits into a short exact sequence (27.5) of vector bundles over P. By Proposition 27.20, there is a one-to-one correspondence between horizontal distributions on P and splittings of the sequence (27.5).

Problems

27.1. Lie bracket of fundamental vector fields
Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let P be a manifold on which G acts on the right. Prove that for $A, B \in \mathfrak{g}$,

$$\underline{[A,B]} = [\underline{A}, \underline{B}].$$

Hence, the map $\sigma\colon \mathfrak{g} \to \mathfrak{X}(P)$, $A \mapsto \underline{A}$ is a Lie algebra homomorphism.

27.2.* Short exact sequence of vector spaces
Prove that if $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is a short exact sequence of finite-dimensional vector spaces, then $\dim B = \dim A + \dim C$.

27.3. Splitting of a short exact sequence
Suppose $0 \to A \xrightarrow{i} B \to C \to 0$ is a short exact sequence of vector bundles over a manifold P. A ***retraction*** of $i\colon A \to B$ is a map $r\colon B \xrightarrow{j} A$ such that $r \circ i = \mathbb{1}_A$. Show that i has a retraction if and only if the sequence has a splitting.

27.4.* The differential of an action
Let $\mu\colon P \times G \to P$ be an action of a Lie group G on a manifold P. For $g \in G$, the tangent space T_gG may be identified with $\ell_{g*}\mathfrak{g}$, where $\ell_g\colon G \to G$ is left multiplication by $g \in G$ and $\mathfrak{g} = T_eG$ is the Lie algebra of G. Hence, an element of the tangent space $T_{(p,g)}(P \times G)$ is of the form $(X_p, \ell_{g*}A)$ for $X_p \in T_pP$ and $A \in \mathfrak{g}$. Prove that the differential

$$\mu_* = \mu_{*,(p,g)}\colon T_{(p,g)}(P \times G) \to T_{pg}P$$

is given by

$$\mu_*(X_p, \ell_{g*}A) = r_{g*}(X_p) + \underline{A}_{pg}.$$

27.5. Fundamental vector field under a trivialization
Let $\phi_\alpha\colon \pi^{-1}U_\alpha \to U_\alpha \times G$

$$\phi_\alpha(p) = \big(\pi(p), g_\alpha(p)\big)$$

be a trivialization of $\pi^{-1}U_{ga}$ in a principal bundle P. Let $A \in \mathfrak{g}$, the Lie algebra of G and $\underline{A}$ the fundamental vector field on P that it induces. Prove that

$$g_{\alpha*}(\underline{A}_p) = \ell_{g_\alpha(p)*}(A) \in T_{g_\alpha(p)}(G).$$

27.6. Trivial principal bundle
Prove that a principal bundle $\pi\colon P \to M$ is trivial if and only if it has a section.

27.7. Pullback of a principal bundle to itself
Prove that if $\pi\colon P \to M$ is a principal bundle, then the pullback bundle $\pi^*P \to P$ is trivial.

27.8. Quotient space of a principal bundle
Let G be a Lie group and H a closed subgroup. Prove that if $\pi P \to M$ is a principal G-bundle, then $P \to P/H$ is a principal H-subbundle.

27.9. Fundamental vector fields
Let N and M be G-manifolds with G acting on the right. If $A \in \mathfrak{g}$ and $f\colon N \to M$ is G-equivariant, then

$$f_*(\underline{A}_{N,q}) = \underline{A}_{M,f(q)}.$$

§28 Connections on a Principal Bundle

Let G be a Lie group with Lie algebra $\mathfrak{g}$. As we saw in the preceding section, on a principal G-bundle $P \to M$, the notion of a vertical tangent vector is well defined, but not that of a horizontal tangent vector. A connection on a principal bundle is essentially the choice of a horizontal complement to the vertical tangent bundle on P. Alternatively, it can be given by a $\mathfrak{g}$-valued 1-form on P. In this section we will study these two equivalent manifestations of a connection:

(i) a smooth right-invariant horizontal distribution on P,
(ii) a smooth G-equivariant $\mathfrak{g}$-valued 1-form ω on P such that on the fundamental vector fields,

$$\omega(\underline{A}) = A \quad \text{for all } A \in \mathfrak{g}. \tag{28.1}$$

Under the identification of $\mathfrak{g}$ with a vertical tangent space, condition (28.1) says that ω restricts to the identity map on vertical vectors.

The correspondence between (i) and (ii) is easy to describe. Given a right-invariant horizontal distribution $\mathcal{H}$ on P, we define a $\mathfrak{g}$-valued 1-form ω on P to be, at each point p, the projection with kernel $\mathcal{H}_p$ from the tangent space to the vertical space. Conversely, given a right-equivariant $\mathfrak{g}$-valued 1-form ω that is the identity on the vertical space at each point $p \in P$, we define a horizontal distribution $\mathcal{H}$ on P to be $\ker \omega_p$ at each $p \in P$.

28.1 Connections on a Principal Bundle

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $\pi\colon P \to M$ be a principal G-bundle. A ***distribution*** on a manifold is a subbundle of the tangent bundle. Recall that a distribution $\mathcal{H}$ on P is ***horizontal*** if it is complementary to the vertical subbundle $\mathcal{V}$ of the tangent bundle TP: for all p in P,

$$T_pP = \mathcal{V}_p \oplus \mathcal{H}_p.$$

Suppose $\mathcal{H}$ is a horizontal distribution on the total space P of a principal G-bundle $\pi\colon P \to M$. For $p \in P$, if $j_p\colon G \to P$ is the map $j_p(g) = p \cdot g$, then the vertical tangent space $\mathcal{V}_p$ can be canonically identified with the Lie algebra $\mathfrak{g}$ via the isomorphism $j_{p*}\colon \mathfrak{g} \to \mathcal{V}_p$ (Proposition 27.18). Let $v\colon T_pP = \mathcal{V}_p \oplus \mathcal{H}_p \to \mathcal{V}_p$ be the projection to the vertical tangent space with kernel $\mathcal{H}_p$. For $Y_p \in T_pP$, $v(Y_p)$ is called the ***vertical component*** of Y_p. (Although the vertical subspace $\mathcal{V}_p$ is intrinsically defined, the notion of the vertical component of a tangent vector depends on the choice of a horizontal complement $\mathcal{H}_p$.) If ω_p is the composite

$$\omega_p := j_{p*}^{-1} \circ v\colon T_pP \overset{v}{\to} \mathcal{V}_p \overset{j_{p*}^{-1}}{\to} \mathfrak{g}, \tag{28.2}$$

then ω is a $\mathfrak{g}$-valued 1-form on P. In terms of ω, the vertical component of $Y_p \in T_pP$ is

$$v(Y_p) = j_{p*}\big(\omega_p(Y_p)\big). \tag{28.3}$$

Theorem 28.1. *If $\mathcal{H}$ is a smooth right-invariant horizontal distribution on the total space P of a principal G-bundle $\pi\colon P \to M$, then the $\mathfrak{g}$-valued 1-form ω on P defined above satisfies the following three properties:*

(i) *for any $A \in \mathfrak{g}$ and $p \in P$, we have $\omega_p(\underline{A}_p) = A$;*
(ii) (G-equivariance) *for any $g \in G$, $r_g^*\omega = (\operatorname{Ad} g^{-1})\omega$;*
(iii) *ω is C^∞.*

Proof. (i) Since $\underline{A}_p$ is already vertical (Proposition 27.17), the projection v leaves it invariant, so

$$\omega_p(\underline{A}_p) = j_{p*}^{-1}\left(v(\underline{A}_p)\right) = j_{p*}^{-1}(\underline{A}_p) = A.$$

(ii) For $p \in P$ and $Y_p \in T_pP$, we need to show

$$\omega_{pg}(r_{g*}Y_p) = (\operatorname{Ad} g^{-1})\omega_p(Y_p).$$

Since both sides are $\mathbb{R}$-linear in Y_p and Y_p is the sum of a vertical and a horizontal vector, we may treat these two cases separately.

If Y_p is vertical, then by Proposition 27.18, $Y_p = \underline{A}_p$ for some $A \in \mathfrak{g}$. In this case

$$\begin{aligned}
\omega_{pg}(r_{g*}\underline{A}_p) &= \omega_{pg}\left(\underline{(\operatorname{Ad} g^{-1})A}_{pg}\right) && \text{(by Proposition 27.13)}\\
&= (\operatorname{Ad} g^{-1})A && \text{(by (i))}\\
&= (\operatorname{Ad} g^{-1})\omega_p(\underline{A}_p) && \text{(by (i) again).}
\end{aligned}$$

If Y_p is horizontal, then by the right-invariance of the horizontal distribution $\mathcal{H}$, so is $r_{g*}Y_p$. Hence,

$$\omega_{pg}(r_{g*}Y_p) = 0 = (\operatorname{Ad} g^{-1})\omega_p(Y_p).$$

(iii) Fix a point $p \in P$. We will show that ω is C^∞ in a neighborhood of p. Let $B_1, \dots, B_\ell$ be a basis for the Lie algebra $\mathfrak{g}$ and $\underline{B_1}, \dots, \underline{B_\ell}$ the associated fundamental vector fields on P. By Proposition 27.11, these vector fields are all C^∞ on P. Since $\mathcal{H}$ is a C^∞ distribution on P, one can find a neighborhood W of p and C^∞ horizontal vector fields $X_1, \dots, X_n$ on W that span $\mathcal{H}$ at every point of W. Then $\underline{B_1}, \dots, \underline{B_\ell}, X_1, \dots, X_n$ is a C^∞ frame for the tangent bundle TP over W. Thus, any C^∞ vector field X on W can be written as a linear combination

$$X = \sum a^i \underline{B_i} + \sum b^j X_j$$

with C^∞ coefficients a^i, b^j on W. By the definition of ω,

$$\omega(X) = \omega\left(\sum a^i \underline{B_i}\right) = \sum a^i B_i.$$

This proves that ω is a C^∞ 1-form on W. □

Note that in this theorem the proof of the smoothness of ω requires only that the horizontal distribution $\mathcal{H}$ be smooth; it does not use the right-invariance of $\mathcal{H}$.

Definition 28.2. An ***Ehresmann connection*** or simply a ***connection*** on a principal G-bundle $P \to M$ is a $\mathfrak{g}$-valued 1-form ω on P satisfying the three properties of Theorem 28.1.

A $\mathfrak{g}$-valued 1-form α on P can be viewed as a map $\alpha\colon TP \to \mathfrak{g}$ from the tangent bundle TP to the Lie algebra $\mathfrak{g}$. Now both TP and $\mathfrak{g}$ are G-manifolds: the Lie group G acts on TP on the right by the differentials of right translations and it acts on $\mathfrak{g}$ on the left by the adjoint representation. By (27.1), $\alpha\colon TP \to \mathfrak{g}$ is G-equivariant if and only if for all $p \in P$, $X_p \in T_pP$, and $g \in G$,

$$\alpha(X_p \cdot g) = g^{-1} \cdot \alpha(X_p),$$

or

$$\alpha(r_{g*}X_p) = (\operatorname{Ad} g^{-1})\alpha(X_p).$$

Thus, $\alpha\colon TP \to \mathfrak{g}$ is G-equivariant if and only if $r_g^*\alpha = (\operatorname{Ad} g^{-1})\alpha$ for all $g \in G$. Condition (ii) of a connection ω on a principal bundle says precisely that ω is G-equivariant as a map from TP to $\mathfrak{g}$.

28.2 Vertical and Horizontal Components of a Tangent Vector

As we noted in Section 27.5, on any principal G-bundle $\pi : P \to M$, the vertical subspace $\mathcal{V}_p$ of the tangent space T_pP is intrinsically defined:

$$\mathcal{V}_p := \ker \pi_* : T_pP \to T_{\pi(p)}M.$$

By Proposition 27.18, the map j_{p*} naturally identifies the Lie algebra $\mathfrak{g}$ of G with the vertical subspace $\mathcal{V}_p$.

In the presence of a horizontal distribution on the total space P of a principal bundle, every tangent vector $Y_p \in T_pP$ decomposes uniquely into the sum of a vertical vector and a horizontal vector:

$$Y_p = v(Y_p) + h(Y_p) \in \mathcal{V}_p \oplus \mathcal{H}_p.$$

These are called, respectively, the ***vertical component*** and ***horizontal component*** of the vector Y_p. As p varies over P, this decomposition extends to a decomposition of a vector field Y on P:

$$Y = v(Y) + h(Y).$$

We often omit the parentheses in $v(Y)$ and $h(Y)$, and write vY and hY instead.

Proposition 28.3. *If $\mathcal{H}$ is a C^∞ horizontal distribution on the total space P of a principal bundle, then the vertical and horizontal components $v(Y)$ and $h(Y)$ of a C^∞ vector field Y on P are also C^∞.*

Proof. Let ω be the $\mathfrak{g}$-valued 1-form associated to the horizontal distribution $\mathcal{H}$ by (28.2). It is C^∞ by Theorem 28.1(iii). In terms of a basis $B_1, \dots, B_\ell$ for $\mathfrak{g}$, we can write $\omega = \sum \omega^i B_i$, where ω^i are C^∞ 1-forms on P. If $Y_p \in T_pP$, then by (28.3) its vertical component $v(Y_p)$ is

$$v(Y_p) = j_{p*}(\omega_p(Y_p)) = j_{p*}\left(\sum \omega_p^i(Y_p)B_i\right) = \sum \omega_p^i(Y_p)(\underline{B_i})_p.$$

As p varies over P,

$$v(Y) = \sum \omega^i(Y)\underline{B_i}.$$

Since ω^i, Y, and $\underline{B_i}$ are all C^∞, so is $v(Y)$. Because $h(Y) = Y - v(Y)$, the horizontal component $h(Y)$ of a C^∞ vector field Y on P is also C^∞. □

On a principal bundle $\pi\colon P \to M$, if $r_g\colon P \to P$ is right translation by $g \in G$, then $\pi \circ r_g = \pi$. It follows that $\pi_* \circ r_{g*} = \pi_*$. Thus, the right translation $r_{g*} : T_pP \to T_{pg}P$ sends a vertical vector to a vertical vector. By hypothesis, $r_{g*}\mathcal{H}_p = \mathcal{H}_{pg}$ and hence the right translation r_{g*} also sends a horizontal vector to a horizontal vector.

Proposition 28.4. *Suppose $\mathcal{H}$ is a smooth right-invariant horizontal distribution on the total space of a principal G-bundle $\pi\colon P \to M$. For each $g \in G$, the right translation r_{g*} commutes with the projections v and h.*

Proof. Any $X_p \in T_pP$ decomposes into vertical and horizontal components:

$$X_p = v(X_p) + h(X_p).$$

Applying r_{g*} to both sides, we get

$$r_{g*}X_p = r_{g*}v(X_p) + r_{g*}h(X_p). \tag{28.4}$$

Since r_{g*} preserves vertical and horizontal subspaces, $r_{g*}v(X_p)$ is vertical and $r_{g*}h(X_p)$ is horizontal. Thus, (28.4) is the decomposition of $r_{g*}X_p$ into vertical and horizontal components. This means for every $X_p \in T_pP$,

$$vr_{g*}(X_p) = r_{g*}v(X_p) \quad \text{and} \quad hr_{g*}(X_p) = r_{g*}h(X_p).$$ □

28.3 The Horizontal Distribution of an Ehresmann Connection

In Section 28.1 we showed that a smooth, right-invariant horizontal distribution on the total space of a principal bundle determines an Ehresmann connection. We now prove the converse.

Theorem 28.5. *If ω is a connection on the principal G-bundle $\pi\colon P \to M$, then $\mathcal{H}_p := \ker \omega_p$, $p \in P$, is a smooth right-invariant horizontal distribution on P.*

Proof. We need to verify three properties:

(i) At each point p in P, the tangent space T_pP decomposes into a direct sum $T_pP = \mathcal{V}_p \oplus \mathcal{H}_p$.

(ii) For $p \in P$ and $g \in G$, $r_{g*}(\mathcal{H}_p) \subset \mathcal{H}_{pg}$.
(iii) $\mathcal{H}$ is a C^∞ subbundle of the tangent bundle TP.

(i) Since $\mathcal{H}_p = \ker \omega_p$, there is an exact sequence

$$0 \to \mathcal{H}_p \to T_pP \overset{\omega_p}{\to} \mathfrak{g} \to 0.$$

The map $j_{p*}\colon \mathfrak{g} \to \mathcal{V}_p \subset T_pP$ provides a splitting of the sequence. By Proposition 27.20, there is a sequence of isomorphisms

$$T_pP \simeq \mathfrak{g} \oplus \mathcal{H}_p \simeq \mathcal{V}_p \oplus \mathcal{H}_p.$$

(ii) Suppose $Y_p \in \mathcal{H}_p = \ker \omega_p$. By the right-equivariance property of an Ehresmann connection,

$$\omega_{pg}(r_{g*}Y_p) = (r_g^*\omega)_p(Y_p) = (\operatorname{Ad} g^{-1})\omega_p(Y_p) = 0.$$

Hence, $r_{g*}Y_p \in \mathcal{H}_{pg}$.

(iii) Let $B_1, \ldots, B_\ell$ be a basis for the Lie algebra $\mathfrak{g}$ of G. Then $\omega = \sum \omega^i B_i$, where $\omega^1, \ldots, \omega^\ell$ are smooth $\mathbb{R}$-valued 1-forms on P and for $p \in P$,

$$\mathcal{H}_p = \bigcap_{i=1}^{\ell} \ker \omega_p^i.$$

Since $\omega_p\colon T_pP \to \mathfrak{g}$ is surjective, $\omega^1, \ldots, \omega^\ell$ are linearly independent at p.

Fix a point $p \in P$ and let $x^1, \ldots, x^m$ be local coordinates near p on P. Then

$$\omega^i = \sum_{j=1}^{m} f_j^i \, dx^j, \quad i = 1, \ldots, \ell$$

for some C^∞ functions f_j^i in a neighborhood of p.

Let $b^1, \ldots, b^m$ be the fiber coordinates of TP near p, i.e., if $v_q \in T_qP$ for q near p, then

$$v_q = \sum b^j \left.\frac{\partial}{\partial x^j}\right|_q.$$

In terms of local coordinates,

$$\begin{aligned}\mathcal{H}_q = \bigcap_{i=1}^{\ell} \ker \omega_q^i &= \{v_q \in T_qP \mid \omega_q^i(v_q) = 0, i = 1, \ldots, \ell\} \\ &= \{(b^1, \ldots, b^m) \in \mathbb{R}^m \mid \sum_{j=1}^{m} f_j^i(q) b^j = 0, i = 1, \ldots, \ell\}.\end{aligned}$$

Let $F^i(q,b) = \sum_{j=1}^m f_j^i(q) b^j$, $i = 1, \ldots, \ell$. Since $\omega^1, \ldots, \omega^\ell$ are linearly independent at p, the Jacobian matrix $[\partial F^i / \partial b^j] = [f_j^i]$, an $\ell \times m$ matrix, has rank ℓ at p. Without loss of generality, we may assume that the first $\ell \times \ell$ block of $[f_j^i(p)]$ has

rank ℓ. Since having maximal rank is an open condition, there is a neighborhood U_p of p on which the first $\ell \times \ell$ block of $[f^i_j]$ has rank ℓ. By the implicit function theorem, on U_p, $b^1, \dots, b^\ell$ are C^∞ functions of $b^{\ell+1}, \dots, b^m$, say

$$\begin{aligned} b^1 &= b^1(b^{\ell+1}, \dots, b^m), \\ &\vdots \\ b^\ell &= b^\ell(b^{\ell+1}, \dots, b^m). \end{aligned}$$

Let

$$\begin{aligned} X_1 &= \sum_{j=1}^{\ell} b^j(1,0,\dots,0)\frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^{\ell+1}} \\ X_2 &= \sum_{j=1}^{\ell} b^j(0,1,0,\dots,0)\frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^{\ell+2}} \\ &\vdots \\ X_{m-\ell} &= \sum_{j=1}^{\ell} b^j(0,0,\dots,1)\frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^m}. \end{aligned}$$

These are C^∞ vector fields on U_p that span $\mathcal{H}_q$ at each point $q \in U_p$. By the subbundle criterion (Theorem 20.4), $\mathcal{H}$ is a C^∞ subbundle of TP. □

28.4 Horizontal Lift of a Vector Field to a Principal Bundle

Suppose $\mathcal{H}$ is a horizontal distribution on a principal bundle $\pi\colon P \to M$. Let X be a vector field on M. For every $p \in P$, because the vertical subspace $\mathcal{V}_p$ is $\ker \pi_*$, the differential $\pi_*\colon T_pP \to T_{\pi(p)}M$ induces an isomorphism

$$\frac{T_pP}{\ker \pi_*} \xrightarrow{\sim} \mathcal{H}_p \xrightarrow{\sim} T_{\pi(p)}M$$

of the horizontal subspace $\mathcal{H}_p$ with the tangent space $T_{\pi(p)}M$. Consequently, there is a unique horizontal vector $\tilde{X}_p \in \mathcal{H}_p$ such that $\pi_*(\tilde{X}_p) = X_{\pi(p)} \in T_{\pi(p)}M$. The vector field $\tilde{X}$ is called the ***horizontal lift*** of X to P.

Proposition 28.6. *If $\mathcal{H}$ is a C^∞ right-invariant horizontal distribution on the total space P of a principal bundle $\pi\colon P \to M$, then the horizontal lift $\tilde{X}$ of a C^∞ vector field X on M is a C^∞ right-invariant vector field on P.*

Proof. Let $x \in M$ and $p \in \pi^{-1}(x)$. By definition, $\pi_*(\tilde{X}_p) = X_x$. If q is any other point of $\pi^{-1}(x)$, then $q = pg$ for some $g \in G$. Since $\pi \circ r_g = \pi$,

$$\pi_*(r_{g*}\tilde{X}_p) = (\pi \circ r_g)_*\tilde{X}_p = \pi_*\tilde{X}_p = X_p.$$

By the uniqueness of the horizontal lift, $r_{g*}\tilde{X}_p = \tilde{X}_{pg}$. This proves the right-invariance of $\tilde{X}$.

We prove the smoothness of $\tilde{X}$ by proving it locally. Let $\{U\}$ be a trivializing open cover for P with trivializations $\phi_U : \pi^{-1}(U) \xrightarrow{\sim} U \times G$. Define

$$Z_{(x,g)} = (X_x, 0) \in T_{(x,g)}(U \times G).$$

Let $\eta : U \times G \to U$ be the projection to the first factor. Then Z is a C^∞ vector field on $U \times G$ such that $\eta_* Z_{(x,g)} = X_x$, and $Y := (\phi_{U*})^{-1} Z$ is a C^∞ vector field on $\pi^{-1}(U)$ such that $\pi_* Y_p = X_{\pi(p)}$. By Proposition 28.3, hY is a C^∞ vector field on $\pi^{-1}(U)$. Clearly it is horizontal. Because $Y_p = v(Y_p) + h(Y_p)$ and $\pi_* v(Y_p) = 0$, we have $\pi_* Y_p = \pi_* h(Y_p) = X_{\pi(p)}$. Thus, hY lifts X over U. By the uniqueness of the horizontal lift, $hY = \widetilde{X}$ over U. This proves that $\tilde{X}$ is a smooth vector field on P. □

28.5 Lie Bracket of a Fundamental Vector Field

If a principal bundle P comes with a connection, then it makes sense to speak of horizontal vector fields on P; these are vector fields all of whose vectors are horizontal.

Lemma 28.7. *Suppose P is a principal bundle with a connection. Let $\underline{A}$ be the fundamental vector field on P associated to $A \in \mathfrak{g}$.*

(i) *If Y is a horizontal vector field on P, then $[\underline{A}, Y]$ is horizontal.*
(ii) *If Y is a right-invariant vector field on P, then $[\underline{A}, Y] = 0$.*

Proof. (i) A local flow for $\underline{A}$ is $\phi_t(p) = pe^{tA} = r_{e^{tA}}(p)$ (Proposition 27.14). By the identification of the Lie bracket with the Lie derivative of vector fields [21, Th. 20.4, p. 225] and the definition of the Lie derivative,

$$[\underline{A}, Y]_p = (\mathcal{L}_{\underline{A}} Y)_p = \lim_{t \to 0} \frac{(r_{e^{-tA}})_* Y_{pe^{tA}} - Y_p}{t}. \tag{28.5}$$

Since right translation preserves horizontality (Theorem 28.5), both $(r_{e^{-tA}})_* Y_{pe^{tA}}$ and Y_p are horizontal vectors. Denote the difference quotient in (28.5) by $c(t)$. For every t near 0 in $\mathbb{R}$, $c(t)$ is in the vector space $\mathcal{H}_p$ of horizontal vectors at p. Therefore, $[\underline{A}, Y]_p = \lim_{t \to 0} c(t) \in \mathcal{H}_p$.

(ii) If Y is right-invariant, then

$$(r_{e^{-tA}})_* Y_{pe^{tA}} = Y_p.$$

In that case, it follows from (28.5) that $[\underline{A}, Y]_p = 0$. □

Problems

28.1. Maurer–Cartan connection
If θ is the Maurer–Cartan form on a Lie group and $\pi_2 : M \times G \to G$ is the projection to the second factor, prove that $\omega := \pi_2^* \theta$ is a connection on the trivial bundle $\pi_1 : M \times G \to M$. It is called the ***Maurer–Cartan connection***.

28.2. Convex linear combinations of connections
Prove that a convex linear combination ω of connections $\omega_1, \dots, \omega_n$ on a principal bundle $\pi\colon P \to M$ is again a connection on P. ($\omega = \sum \lambda_i \omega_i$, $\sum \lambda_i = 1$, $\lambda_i \geq 0$.)

28.3. Pullback of a connection
Let $\pi_Q\colon Q \to N$ and $\pi_P\colon P \to M$ be principal G-bundles, and let $(\bar{f}\colon Q \to P, f\colon N \to M)$ be a morphism of principal bundles. Prove that if θ is a connection on P, then $\bar{f}^*\theta$ is a connection on Q.

§29 Horizontal Distributions on a Frame Bundle

In this section we will explain the process by which a connection ∇ on a vector bundle E over a manifold M gives rise to a smooth right-invariant horizontal distribution on the associated frame bundle $\mathrm{Fr}(E)$. This involves a sequence of steps. A connection on the vector bundle E induces a covariant derivative on sections of the vector bundle along a curve. Parallel sections along the curve are those whose derivative vanishes. Just as for tangent vectors in Section 14, starting with a frame e_x for the fiber of the vector bundle at the initial point x of a curve, there is a unique way to parallel translate the frame along the curve. In terms of the frame bundle $\mathrm{Fr}(E)$, what this means is that every curve in M has a unique lift to $\mathrm{Fr}(E)$ starting at e_x representing parallel frames along the curve. Such a lift is called a ***horizontal lift***. The initial vector at e_x of a horizontal lift is a ***horizontal vector*** at e_x. The horizontal vectors at a point of $\mathrm{Fr}(E)$ form a subspace of the tangent space $T_{e_x}\big(\mathrm{Fr}(E)\big)$. In this way we obtain a horizontal distribution on the frame bundle. We show that this horizontal distribution on $\mathrm{Fr}(E)$ arising from a connection on the vector bundle E is smooth and right-invariant. It therefore corresponds to a connection ω on the principal bundle $\mathrm{Fr}(E)$. We then show that ω pulls back under a section e of $\mathrm{Fr}(E)$ to the connection matrix ω_e of the connection ∇ relative to the frame e on an open set U.

29.1 Parallel Translation in a Vector Bundle

In Section 14 we defined parallel translation of a tangent vector along a curve in a manifold with an affine connection. In fact, the same development carries over to an arbitrary vector bundle $\eta\colon E \to M$ with a connection ∇.

Let $c\colon [a,b] \to M$ be a smooth curve in M. Instead of vector fields along the curve c, we consider smooth sections of the pullback bundle c^*E over $[a,b]$. These are called ***smooth sections of the vector bundle*** E ***along the curve*** c. We denote by $\Gamma(c^*E)$ the vector space of smooth sections of E along the curve c. If $E = TM$ is the tangent bundle of a manifold M, then an element of $\Gamma(c^*TM)$ is simply a vector field along the curve c in M. Just as in Theorem 13.1, there is a unique $\mathbb{R}$-linear map

$$\frac{D}{dt}\colon \Gamma(c^*E) \to \Gamma(c^*E),$$

called the ***covariant derivative*** corresponding to ∇, such that

(i) (Leibniz rule) for any C^∞ function f on the interval $[a,b]$,

$$\frac{D(fs)}{dt} = \frac{df}{dt}s + f\frac{Ds}{dt};$$

(ii) if s is induced from a global section $\tilde{s} \in \Gamma(M,E)$ in the sense that $s(t) = \tilde{s}\big(c(t)\big)$, then

$$\frac{Ds}{dt}(t) = \nabla_{c'(t)}\tilde{s}.$$

Definition 29.1. A section $s \in \Gamma(c^*E)$ is ***parallel along a curve*** $c\colon [a,b] \to M$ if $Ds/dt \equiv 0$ on $[a,b]$.

As in Section 14.5, the equation $Ds/dt \equiv 0$ for a section s to be parallel is equivalent to a system of linear first-order ordinary differential equations. Suppose $c\colon [a,b] \to M$ maps into a framed open set $(U, e_1, \dots, e_r)$ for E. Then $s \in \Gamma(c^*E)$ can be written as

$$s(t) = \sum s^i(t) e_{i,c(t)}.$$

By properties (i) and (ii) of the covariant derivative,

$$\begin{aligned} \frac{Ds}{dt} &= \sum_i \frac{ds^i}{dt} e_i + \sum_j s^j \frac{D}{dt} e_{j,c(t)} \\ &= \sum_i \frac{ds^i}{dt} e_i + \sum_j s^j \nabla_{c'(t)} e_j \\ &= \sum_i \frac{ds^i}{dt} e_i + \sum_{i,j} s^j \omega^i_j\big(c'(t)\big) e_i. \end{aligned}$$

Hence, $Ds/dt \equiv 0$ if and only if

$$\frac{ds^i}{dt} + \sum_j s^j \omega^i_j\big(c'(t)\big) = 0 \text{ for all } i.$$

This is a system of linear first-order differential equations. By the existence and uniqueness theorems of differential equations, it has a solution on a small interval about a give point t_0 and the solution is uniquely determined by its value at t_0. Thus, a parallel section is uniquely determined by its value at a point. If $s \in \Gamma(c^*E)$ is a parallel section of the pullback bundle c^*E, we say that $s(b)$ is the ***parallel transport*** of $s(a)$ along $c\colon [a,b] \to M$. The resulting map: $E_{c(a)} \to E_{c(b)}$ is called ***parallel translation*** from $E_{c(a)}$ to $E_{c(b)}$.

Theorem 29.2. *Let $\eta : E \to M$ be a C^∞ vector bundle with a connection ∇ and let $c\colon [a,b] \to M$ be a smooth curve in M. There is a unique parallel translation $\varphi_{a,b}$ from $E_{c(a)}$ to $E_{c(b)}$ along c. This parallel translation $\varphi_{a,b}\colon E_{c(a)} \to E_{c(b)}$ is a linear isomorphism.*

The proof is similar to that of Theorem 14.14.

A ***parallel frame along*** the curve $c\colon [a,b] \to M$ is a collection of parallel sections $\big(e_1(t), \dots, e_r(t)\big)$, $t \in [a,b]$, such that for each t, the elements $e_1(t), \dots, e_r(t)$ form a basis for the vector space $E_{c(t)}$.

Let $\pi\colon \operatorname{Fr}(E) \to M$ be the frame bundle of the vector bundle $\eta : E \to M$. A curve $\tilde{c}(t)$ in $\operatorname{Fr}(E)$ is called a ***lift*** of the curve $c(t)$ in M if $c(t) = \pi(\tilde{c}(t))$. It is a ***horizontal lift*** if in addition $\tilde{c}(t)$ is a parallel frame along c.

Restricting the domain of the curve c to the interval $[a,t]$, we obtain from Theorem 29.2 that parallel translation is a linear isomorphism of $E_{c(a)}$ with $E_{c(t)}$. Thus, if

a collection of parallel sections $\big(s_1(t),\dots,s_r(t)\big) \in \Gamma(c^*E)$ forms a basis at one time t, then it forms a basis at every time $t \in [a,b]$. By Theorem 29.2, for every smooth curve $c\colon [a,b] \to M$ and ordered basis $(s_{1,0},\dots,s_{r,0})$ for $E_{c(a)}$, there is a unique parallel frame along c whose value at a is $(s_{1,0},\dots,s_{r,0})$. In terms of the frame bundle $\mathrm{Fr}(E)$, this shows the existence and uniqueness of a horizontal lift with a specified initial point in $\mathrm{Fr}(E)$ of a curve $c(t)$ in M.

29.2 Horizontal Vectors on a Frame Bundle

On a general principal bundle vertical vectors are intrinsically defined, but horizontal vectors are not. We will see shortly that a connection on a vector bundle E over a manifold M determines a well-defined horizontal distribution on the frame bundle $\mathrm{Fr}(E)$. The elements of the horizontal distribution are the horizontal vectors. Thus, the notion of a horizontal vector on the frame bundle $\mathrm{Fr}(E)$ depends on a connection on E.

Definition 29.3. Let $E \to M$ be a vector bundle with a connection ∇, $x \in M$, and $e_x \in \mathrm{Fr}(E_x)$. A tangent vector $v \in T_{e_x}(\mathrm{Fr}(E))$ is said to be ***horizontal*** if there is a curve $c(t)$ through x in M such that v is the initial vector $\tilde{c}'(0)$ of the unique horizontal lift of $\tilde{c}(t)$ of $c(t)$ to $\mathrm{Fr}(E)$ starting at e_x.

Our goal now is to show that the horizontal vectors at a point e_x of the frame bundle form a vector subspace of the tangent space $T_{e_x}\big(\mathrm{Fr}(E)\big)$. To this end we will derive an explicit formula for $\tilde{c}'(0)$ in terms of a local frame for E. Suppose $c\colon [0,b] \to M$ is a smooth curve with initial point $c(0) = x$, and $\tilde{c}(t)$ is its unique horizontal lift to $\mathrm{Fr}(E)$ with initial point $e_x = (e_{1,0},\dots,e_{r,0})$. Let s be a frame for E over a neighborhood U of x with $s(x) = e_x$. Then $s(c(t))$ is a lift of $c(t)$ to $\mathrm{Fr}(E)$ with initial point e_x, but of course it is not necessarily a horizontal lift (see Figure 29.1). For any $t \in [0,b]$, we have two ordered bases $s(c(t))$ and $\tilde{c}(t)$ for $E_{c(t)}$, so there is a smooth matrix $a(t) \in \mathrm{GL}(r,\mathbb{R})$ such that $s(c(t)) = \tilde{c}(t)a(t)$. At $t = 0$, $s(c(0)) = e_x = \tilde{c}(0)$, so that $a(0) = I$, the identity matrix in $\mathrm{GL}(r,\mathbb{R})$.

Lemma 29.4. *In the notation above, let $s_*\colon T_x(M) \to T_{e_x}(\mathrm{Fr}(E))$ be the differential of s and $\underline{a'(0)}$ the fundamental vector field on $\mathrm{Fr}(E)$ associated to $a'(0) \in \mathfrak{gl}(r,\mathbb{R})$. Then*

$$s_*\big(c'(0)\big) = \tilde{c}'(0) + \underline{a'(0)}_{e_x}.$$

Proof. Let $P = \mathrm{Fr}(E)$ and $G = \mathrm{GL}(r,\mathbb{R})$, and let $\mu\colon P \times G \to P$ be the right action of G on P. Then

$$s\big(c(t)\big) = \tilde{c}(t)a(t) = \mu\big(\tilde{c}(t),a(t)\big), \tag{29.1}$$

with $c(0) = x$, $\tilde{c}(0) = e_x$, and $a(0) =$ the identity matrix I. Differentiating (29.1) with respect to t and evaluating at 0 gives

$$s_*\big(c'(0)\big) = \mu_{*,(\tilde{c}(0),a(0))}\big(\tilde{c}'(0),a'(0)\big).$$

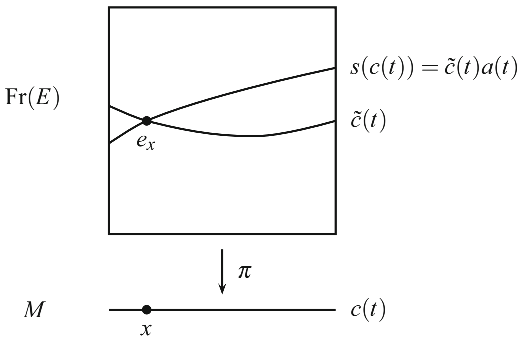


Fig. 29.1. Two liftings of a curve

By the formula for the differential of an action (Problem 27.4),

$$s_*\big(c'(0)\big) = r_{a(0)*}\tilde{c}'(0) + \underline{a'(0)}_{\tilde{c}(0)} = \tilde{c}'(0) + \underline{a'(0)}_{e_x}. \qquad \square$$

Lemma 29.5. *Let $E \to M$ be a vector bundle with a connection ∇. Suppose $s = (s_1, \dots, s_r)$ is a frame for E over an open set U, $\tilde{c}(t)$ a parallel frame over a curve $c(t)$ in U with $\tilde{c}(0) = s(c(0))$, and $a(t)$ the curve in $\mathrm{GL}(r,\mathbb{R})$ such that $s(c(t)) = \tilde{c}(t)a(t)$. If $\omega_s = [\omega^i_j]$ is the connection matrix of ∇ with respect to the frame $(s_1, \dots, s_r)$ over U, then $a'(0) = \omega_s(c'(0))$.*

Proof. Label $c(0) = x$ and $\tilde{c}_i(0) = s_i\big(c(0)\big) = e_{i,x}$. By the definition of the connection matrix,

$$\nabla_{c'(0)} s_j = \sum \omega^i_j\big(c'(0)\big) s_i\big(c(0)\big) = \sum \omega^i_j\big(c'(0)\big) e_{i,x}. \tag{29.2}$$

On the other hand, by the defining properties of the covariant derivative (Section 29.1),

$$\begin{aligned}\nabla_{c'(t)} s_j &= \frac{D(s_j \circ c)}{dt}(t) = \frac{D}{dt}\sum \tilde{c}_i(t) a^i_j(t) \\ &= \sum (a^i_j)'(t)\tilde{c}_i(t) + \sum a^i_j(t)\frac{D\tilde{c}_i}{dt}(t) \\ &= \sum (a^i_j)'(t)\tilde{c}_i(t) \qquad (\text{since } D\tilde{c}_i/dt \equiv 0).\end{aligned}$$

Setting $t = 0$ gives

$$\nabla_{c'(0)} s_j = \sum (a^i_j)'(0) e_{i,x}. \tag{29.3}$$

Equating (29.2) and (29.3), we obtain $(a^i_j)'(0) = \omega^i_j(c'(0))$. $\square$

Thus, Lemma 29.4 for the horizontal lift of $c'(0)$ can be rewritten in the form

$$\tilde{c}'(0) = s_*\big(c'(0)\big) - \underline{a'(0)}_{e_x} = s_*\big(c'(0)\big) - \underline{\omega_s\big(c'(0)\big)}_{e_x}. \tag{29.4}$$

Proposition 29.6. *Let $\pi\colon E \to M$ be a smooth vector bundle with a connection over a manifold M of dimension n. For $x \in M$ and e_x an ordered basis for the fiber E_x, the subset $\mathcal{H}_{e_x}$ of horizontal vectors in the tangent space $T_{e_x}(\mathrm{Fr}(E))$ is a vector space of dimension n, and $\pi_*\colon \mathcal{H}_{e_x} \to T_xM$ is a linear isomorphism.*

Proof. In formula (29.4), $\omega_s(c'(0))$ is $\mathbb{R}$-linear in its argument $c'(0)$ because ω_s is a 1-form at $c(0)$. The operation $A \mapsto \underline{A}_{e_x}$ of associating to a matrix $A \in \mathfrak{gl}(r,\mathbb{R})$ a tangent vector $\underline{A}_{e_x} \in T_{e_x}\big(\mathrm{Fr}(E)\big)$ is $\mathbb{R}$-linear by (27.3). Hence, formula (29.4) shows that the map

$$\begin{aligned}\phi\colon T_xM &\to T_{e_x}\big(\mathrm{Fr}(E)\big),\\ c'(0) &\mapsto \tilde{c}'(0)\end{aligned}$$

is $\mathbb{R}$-linear. As the image of a vector space T_xM under a linear map, the set $\mathcal{H}_{e_x}$ of horizontal vectors $\tilde{c}'(0)$ at e_x is a vector subspace of $T_{e_x}\big(\mathrm{Fr}(E)\big)$.

Since $\pi\big(\tilde{c}(t)\big) = c(t)$, taking the derivative at $t = 0$ gives $\pi_*\big(\tilde{c}'(0)\big) = c'(0)$, so π_* is a left inverse to the map ϕ. This proves that $\phi\colon T_xM \to T_{e_x}\big(\mathrm{Fr}(E)\big)$ is injective. Its image is by definition $\mathcal{H}_{e_x}$. It follows that $\phi\colon T_xM \to \mathcal{H}_{e_x}$ is an isomorphism with inverse $\pi_*\colon \mathcal{H}_{e_x} \to T_{e_x}M$. □

29.3 Horizontal Lift of a Vector Field to a Frame Bundle

We have learned so far that a connection on a vector bundle $E \to M$ defines a horizontal subspace $\mathcal{H}_p$ of the tangent space T_pP at each point p of the total space of the frame bundle $\pi\colon P = \mathrm{Fr}(E) \to M$. The horizontal subspace $\mathcal{H}_p$ has the same dimension as M. The vertical subspace $\mathcal{V}_p$ of T_pP is the kernel of the surjection $\pi_*\colon T_pP \to T_{\pi(p)}M$; as such, $\dim \mathcal{V}_p = \dim T_pP - \dim M$. Hence, $\mathcal{V}_p$ and $\mathcal{H}_p$ have complementary dimensions in T_pP. Since $\pi_*(\mathcal{V}_p) = 0$ and $\pi_*\colon \mathcal{H}_p \to T_{\pi(p)}M$ is an isomorphism, $\mathcal{V}_p \cap \mathcal{H}_p = 0$. It follows that there is a direct sum decomposition

$$T_p(\mathrm{Fr}(E)) = \mathcal{V}_p \oplus \mathcal{H}_p. \tag{29.5}$$

Our goal now is to show that as p varies in P, the subset $\mathcal{H} := \bigcup_{p\in P} \mathcal{H}_p$ of the tangent bundle TP defines a C^∞ horizontal distribution on P in the sense of Section 27.6.

Since $\pi_{*,p}\colon \mathcal{H}_p \to T_{\pi(p)}M$ is an isomorphism for each $p \in P$, if X is a vector field on M, then there is a unique vector field $\tilde{X}$ on P such that $\tilde{X}_p \in \mathcal{H}_p$ and $\pi_{*,p}(\tilde{X}_p) = X_{\pi(p)}$. The vector field $\tilde{X}$ is called the ***horizontal lift*** of X to the frame bundle P.

Since every tangent vector $X_x \in T_xM$ is the initial vector $c'(0)$ of a curve c, formula (29.4) for the horizontal lift of a tangent vector can be rewritten in the following form.

Lemma 29.7 (Horizontal lift formula). *Suppose ∇ is a connection on a vector bundle $E \to M$ and ω_s is its connection matrix on a framed open set (U,s). For $x \in U$, $p = s(x) \in \mathrm{Fr}(E)$, and $X_x \in T_xM$, let $\tilde{X}_p$ be the horizontal lift of X_x to p in $\mathrm{Fr}(E)$. Then*

$$\tilde{X}_p = s_{*,x}(X_x) - \underline{\omega_s(X_x)}_p.$$

Proposition 29.8. *Let $E \to M$ be a C^∞ rank r vector bundle with a connection and $\pi\colon \operatorname{Fr}(E) \to M$ its frame bundle. If X is a C^∞ vector field on M, then its horizontal lift $\tilde{X}$ to $\operatorname{Fr}(E)$ is a C^∞ vector field.*

Proof. Let $P = \operatorname{Fr}(E)$ and $G = \operatorname{GL}(r,\mathbb{R})$. Since the question is local, we may assume that the bundle P is trivial, say $P = M \times G$. By the right invariance of the horizontal distribution,

$$\tilde{X}_{(x,a)} = r_{a*}\tilde{X}_{(x,1)}. \tag{29.6}$$

Let $s\colon M \to P = M \times G$ be the section $s(x) = (x,1)$. By the horizontal lift formula (Lemma 29.7),

$$\tilde{X}_{(x,1)} = s_{*,x}(X_x) - \underline{\omega_s(X_x)}_{(x,1)}. \tag{29.7}$$

Let $p = (x,a) \in P$ and let f be a C^∞ function on P. We will prove that $\tilde{X}_p f$ is C^∞ as a function of p. By (29.6) and (29.7),

$$\tilde{X}_p f = r_{a*}s_{*,x}(X_x)f - r_{a*}\underline{\omega_s(X_x)}_{(x,1)}f, \tag{29.8}$$

so it suffices to prove separately that $\big(r_{a*}(s_{*,x}X_x)\big)f$ and $\big(r_{a*}\underline{\omega_s(X_x)}_{(x,1)}\big)f$ are C^∞ functions on P.

The first term is

$$\begin{aligned}
\big(r_{a*}s_{*,x}(X_x)\big)f &= X_x(f \circ r_a \circ s) \\
&= X(f \circ r_a \circ s)\big(\pi(p)\big) \\
&= X\big(f\big(s(\pi(p))a\big)\big) = X\big(f\big(\mu(s(\pi(p)),a)\big)\big) \\
&= X\big(f\big(\mu(s(\pi(p)),\pi_2(p))\big)\big),
\end{aligned} \tag{29.9}$$

where $\mu\colon P \times G \to P$ is the action of G on P and $\pi_2\colon P = M \times G \to G$ is the projection $\pi_2(p) = \pi_2(x,a) = a$. The formula (29.9) expresses $\big(r_{a*}s_{*,x}(X_x)\big)f$ as a C^∞ function on P.

By the right equivariance of the connection form ω_s, in (29.8) the second term can be rewritten as

$$\begin{aligned}
r_{a*}\underline{\omega_s(X_x)}_{(x,1)}f &= \underline{(\operatorname{Ad} a^{-1})\omega_s(X_x)}_{(x,a)}f \\
&= \underline{\big(\operatorname{Ad}\pi_2(p)^{-1}\big)\omega_s\big(X_{\pi(p)}\big)}_p f,
\end{aligned}$$

where $\big(\operatorname{Ad}\pi_2(p)^{-1}\big)\omega_s(X_{\pi(p)})$ is a C^∞ function: $P \to \mathfrak{gl}(r,\mathbb{R})$ that we will denote by $A(p)$. The problem now is to show that $p \mapsto \underline{A(p)}_p f$ is a C^∞ function of p.

Let $\mu\colon P \times G \to P$ be the right action of $G = \operatorname{GL}(r,\mathbb{R})$ on $P = \operatorname{Fr}(E)$. Then

$$\underline{A(p)}_p f = \frac{d}{dt}\bigg|_{t=0} f(p \cdot e^{tA(p)}) = \frac{d}{dt}\bigg|_{t=0} f(\mu(p, e^{tA(p)})).$$

Since f, μ, A, and the exponential map are all C^∞ functions, $\underline{A(p)}_p f$ is a C^∞ function of p. Thus, $\tilde{X}_p f$ in (29.8) is a C^∞ function of p. This proves that $\tilde{X}$ is a C^∞ vector field on P. □

Theorem 29.9. *A connection ∇ on a smooth vector bundle $E \to M$ defines a C^∞ distribution $\mathcal{H}$ on the frame bundle $\pi\colon P = \mathrm{Fr}(E) \to M$ such that at any $p \in P$,*

(i) $T_pP = \mathcal{V}_p \oplus \mathcal{H}_p$;
(ii) $r_{g*}(\mathcal{H}_p) = \mathcal{H}_{pg}$ *for any* $g \in G = \mathrm{GL}(r,\mathbb{R})$,

where $r_g\colon P \to P$ is the right action of G on P.

Proof. To prove that $\mathcal{H}$ is a C^∞ subbundle of TP, let U be a coordinate open set in M and $s_1,\dots,s_n$ a C^∞ frame on U. By Proposition 29.8 the horizontal lifts $\widetilde{s_1},\dots,\widetilde{s_n}$ are C^∞ vector fields on $\tilde{U} := \pi^{-1}(U)$. Moreover, for each $p \in \tilde{U}$, since $\pi_{*,p}\colon \mathcal{H}_p \to T_{\pi(p)}M$ is an isomorphism, $(\widetilde{s_1})_p,\dots,(\widetilde{s_n})_p$ form a basis for $\mathcal{H}_p$. Thus, over $\tilde{U}$ the C^∞ sections $\widetilde{s_1},\dots,\widetilde{s_n}$ of TP span $\mathcal{H}$. By Theorem 20.4, this proves that $\mathcal{H}$ is a C^∞ subbundle of TP.

Equation (29.5) establishes (i).

As for (ii), let $\tilde{c}'(0) \in \mathcal{H}_p$, where $c(t)$ is a curve in M and $\tilde{c}(t) = [v_1(t)\ \cdots\ v_r(t)]$ is its horizontal lift to P with initial point p. Here we are writing a frame as a row vector so that the group action is simply matrix multiplication on the right. For any $g = [g^i_j] \in \mathrm{GL}(r,\mathbb{R})$,

$$\tilde{c}(t)g = \left[\sum g^i_1 v_i(t)\ \cdots\ \sum g^i_r v_i(t)\right].$$

Since $Dv_i/dt \equiv 0$ by the horizontality of v_i and g^i_j are constants, $D(\sum g^i_j v_i)/dt \equiv 0$. Thus, $\tilde{c}(t)g$ is the horizontal lift of $c(t)$ with initial point $\tilde{c}(0)g$. It has initial tangent vector

$$\left.\frac{d}{dt}\right|_{t=0} \tilde{c}(t)g = r_{g*}\tilde{c}'(0) \in \mathcal{H}_{pg}.$$

This proves that $r_{g*}\mathcal{H}_p \subset \mathcal{H}_{pg}$. Because $r_{g*}\colon \mathcal{H}_p \to \mathcal{H}_{pg}$ has a two-sided inverse $r_{g^{-1}*}$, it is bijective. In particular, $r_{g*}\mathcal{H}_p = \mathcal{H}_{pg}$. □

29.4 Pullback of a Connection on a Frame Bundle Under a Section

Recall that a connection ∇ on a vector bundle E can be represented on a framed open set $(U, e_1,\dots,e_r)$ for E by a connection matrix ω_e depending on the frame. Such a frame $e = (e_1,\dots,e_r)$ is in fact a section $e\colon U \to \mathrm{Fr}(E)$ of the frame bundle. We now use the horizontal lift formula (Lemma 29.7) to prove that the Ehresmann connection ω on the frame bundle $\mathrm{Fr}(E)$ determined by ∇ pulls back under the section e to the connection matrix ω_e.

Theorem 29.10. *Let ∇ be a connection on a vector bundle $E \to M$ and let ω be the Ehresmann connection on the frame bundle $\mathrm{Fr}(E)$ determined by ∇. If $e = (e_1,\dots,e_r)$ is a frame for E over an open set U, viewed as a section $e\colon U \to \mathrm{Fr}(E)|_U$, and ω_e is the connection matrix of ∇ relative to the frame e, then $\omega_e = e^*\omega$.*

Proof. Let $x \in U$ and $p = e(x) \in \operatorname{Fr}(E)$. Suppose X_x is a tangent vector to M at x. If we write $\omega_{e,x}$ for the value of the connection matrix ω_e at the point $x \in U$, then $\omega_{e,x}$ is an $r \times r$ matrix of 1-forms at x and $\omega_{e,x}(X_x)$ is an $r \times r$ matrix of real numbers, i.e., an element of the Lie algebra $\mathfrak{gl}(r, \mathbb{R})$. The corresponding fundamental vector field on $\operatorname{Fr}(E)$ is $\underline{\omega_{e,x}(X_x)}$. By Lemma 29.7, the horizontal lift of X_x to $p \in \operatorname{Fr}(E)$ is

$$\tilde{X}_p = e_* X_x - \underline{\omega_{e,x}(X_x)}_p .$$

Applying the Ehresmann connection ω_p to both sides of this equation, we get

$$\begin{aligned} 0 = \omega_p(\tilde{X}_p) &= \omega_p(e_* X_x) - \omega_p\left(\underline{\omega_{e,x}(X_x)}_p\right) \\ &= (e^*\omega_p)(X_x) - \omega_{e,x}(X_x) \qquad \text{(by Theorem 28.1(i)).} \end{aligned}$$

Since this is true for all $X_x \in T_x M$,

$$e^*\omega_p = (e^*\omega)_x = \omega_{e,x}. \qquad \square$$

§30 Curvature on a Principal Bundle

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Associated to a connection ω on a principal G-bundle is a $\mathfrak{g}$-valued 2-form Ω called its curvature. The definition of the curvature is suggested by the second structural equation for a connection ∇ on a vector bundle E. Just as the connection form ω on the frame bundle $\operatorname{Fr}(E)$ pulls back by a section e of $\operatorname{Fr}(E)$ to the connection matrix ω_e of ∇ with respect to the frame e, so the curvature form Ω on the frame bundle $\operatorname{Fr}(E)$ pulls back by e to the curvature matrix Ω_e of ∇ with respect to e. Thus, the curvature form Ω on the frame bundle is an intrinsic object of which the curvature matrices Ω_e are but local manifestations.

30.1 Curvature Form on a Principal Bundle

By Theorem 11.1 if ∇ is a connection on a vector bundle $E \to M$, then its connection and curvature matrices ω_e and Ω_e on a framed open set $(U, e) = (U, e_1, \ldots, e_r)$ are related by the second structural equation (Theorem 11.1)

$$\Omega_e = d\omega_e + \omega_e \wedge \omega_e.$$

In terms of the Lie bracket of matrix-valued forms (see (21.12)), this can be rewritten as

$$\Omega_e = d\omega_e + \frac{1}{2}[\omega_e, \omega_e].$$

An Ehresmann connection on a principal bundle is Lie algebra-valued. In a general Lie algebra, the wedge product is not defined, but the Lie bracket is always defined. This strongly suggests the following definition for the curvature of an Ehresmann connection on a principal bundle.

Definition 30.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose ω is an Ehresmann connection on a principal G-bundle $\pi\colon P \to M$. Then the ***curvature*** of the connection ω is the $\mathfrak{g}$-valued 2-form

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega].$$

Recall that frames for a vector bundle E over an open set U are sections of the frame bundle $\operatorname{Fr}(E)$. Let ω be the connection form on the frame bundle $\operatorname{Fr}(E)$ determined by a connection ∇ on E. In the same way that ω pulls back by sections of $\operatorname{Fr}(E)$ to connection matrices, the curvature form Ω of the connection ω on $\operatorname{Fr}(E)$ pulls back by sections to curvature matrices.

Proposition 30.2. *If ∇ is a connection on a vector bundle $E \to M$ and ω is the associated Ehresmann connection on the frame bundle* $\operatorname{Fr}(E)$*, then the curvature matrix Ω_e relative to a frame $e = (e_1, \ldots, e_r)$ for E over an open set U is the pullback $e^*\Omega$ of the curvature Ω on the frame bundle* $\operatorname{Fr}(E)$.

Proof.

$$\begin{aligned}
e^*\Omega &= e^*d\omega + \frac{1}{2}e^*[\omega,\omega] \\
&= de^*\omega + \frac{1}{2}[e^*\omega, e^*\omega] && (e^* \text{ commutes with } d \text{ and } [\ ,\ ] \text{ by Proposition 21.8}) \\
&= d\omega_e + \frac{1}{2}[\omega_e,\omega_e] && (\text{by Theorem 29.10}) \\
&= \Omega_e. && (\text{by the second structural equation})
\end{aligned}$$

□

30.2 Properties of the Curvature Form

Now that we have defined the curvature of a connection on a principal G-bundle $\pi\colon P \to M$, it is natural to study some of its properties. Like a connection form, the curvature form Ω is equivariant with respect to right translation on P and the adjoint representation on $\mathfrak{g}$. However, unlike a connection form, a curvature form is horizontal in the sense that it vanishes as long as one argument is vertical. In this respect it acts almost like the opposite of a connection form, which vanishes on horizontal vectors.

Lemma 30.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\pi\colon P \to M$ a principal G-bundle with a connection ω. Fix a point $p \in P$.*

(i) *Every vertical vector $X_p \in T_pP$ can be extended to a fundamental vector field $\underline{A}$ on P for some $A \in \mathfrak{g}$.*
(ii) *Every horizontal vector $Y_p \in T_pP$ can be extended to the horizontal lift $\tilde{B}$ of a C^∞ vector field B on M.*

Proof. (i) By the surjectivity of $j_{p*}\colon \mathfrak{g} \to \mathcal{V}_p$ (Proposition 27.18) and Equation (27.3),

$$X_p = j_{p*}(A) = \underline{A}_p$$

for some $A \in \mathfrak{g}$. Then the fundamental vector field $\underline{A}$ on P extends X_p.

(ii) Let $x = \pi(p)$ in M and let B_x be the projection $\pi_*(Y_p) \in T_xM$ of the vector Y_p. We can extend B_x to a smooth vector field B on M. The horizontal lift $\tilde{B}$ of B extends Y_p on P. □

By Proposition 28.6, such a horizontal lift $\tilde{B}$ is necessarily right-invariant.

Theorem 30.4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose $\pi\colon P \to M$ is a principal G-bundle, ω a connection on P, and Ω the curvature form of ω.*

(i) (Horizontality) *For $p \in P$ and $X_p, Y_p \in T_pP$,*

$$\Omega_p(X_p,Y_p) = (d\omega)_p(hX_p, hY_p). \tag{30.1}$$

(ii) (G-equivariance) *For $g \in G$, we have $r_g^*\Omega = (\operatorname{Ad} g^{-1})\Omega$.*
(iii) (Second Bianchi identity) $d\Omega = [\Omega, \omega]$.

Proof. (i) Since both sides of (30.1) are linear in X_p and in Y_p, we may decompose X_p and Y_p into vertical and horizontal components, and so it suffices to check the equation for vertical and horizontal vectors only. There are three cases.

Case 1. Both X_p and Y_p are horizontal. Then

$$\begin{aligned}
\Omega_p(X_p,Y_p) &= (d\omega)_p(X_p,Y_p) + \frac{1}{2}[\omega_p,\omega_p](X_p,Y_p) && \text{(definition of } \Omega) \\
&= (d\omega)_p(X_p,Y_p) \\
&\quad + \frac{1}{2}\Big([\omega_p(X_p),\omega_p(Y_p)] - [\omega_p(Y_p),\omega_p(X_p)]\Big) \\
&= (d\omega)_p(X_p,Y_p) && (\omega_p(X_p)=0) \\
&= (d\omega)_p(hX_p,hY_p). && (X_p,\ Y_p \text{ horizontal})
\end{aligned}$$

Case 2. One of X_p and Y_p is horizontal; the other is vertical. Without loss of generality, we may assume X_p vertical and Y_p horizontal. Then $[\omega_p,\omega_p](X_p,Y_p) = 0$ as in Case 1.

By Lemma 30.3 the vertical vector X_p extends to a fundamental vector field $\underline{A}$ on P and the horizontal vector Y_p extends to a right-invariant horizontal vector field $\tilde{B}$ on P. By the global formula for the exterior derivative (Problem 21.8)

$$d\omega(\underline{A},\tilde{B}) = \underline{A}(\omega(\tilde{B})) - \tilde{B}(\omega(\underline{A})) - \omega([\underline{A},\tilde{B}]).$$

On the right-hand side, $\omega(\tilde{B}) = 0$ because $\tilde{B}$ is horizontal, and $\tilde{B}\omega(\underline{A}) = \tilde{B}A = 0$ because A is a constant function on P. Being the bracket of a fundamental and a horizontal vector field, $[\underline{A},\tilde{B}]$ is horizontal by Lemma 28.7, and therefore $\omega([\underline{A},\tilde{B}]) = 0$. Hence, the left-hand side of (30.1) becomes

$$\Omega_p(X_p,Y_p) = (d\omega)_p(\underline{A}_p,\tilde{B}_p) = 0.$$

The right-hand side of (30.1) is also zero because $hX_p = 0$.

Case 3. Both X_p and Y_p are vertical. As in Case 2, we can write $X_p = \underline{A}_p$ and $Y_p = \underline{B}_p$ for some $A,B \in \mathfrak{g}$. We have thus extended the vertical vectors X_p and Y_p to fundamental vector fields $X = \underline{A}$ and $Y = \underline{B}$ on P. By the definition of curvature,

$$\begin{aligned}
\Omega(X,Y) &= \Omega(\underline{A},\underline{B}) \\
&= d\omega(\underline{A},\underline{B}) + \frac{1}{2}\big([\omega(\underline{A}),\omega(\underline{B})] - [\omega(\underline{B}),\omega(\underline{A})]\big) \\
&= d\omega(\underline{A},\underline{B}) + [A,B].
\end{aligned} \tag{30.2}$$

In this sum the first term is

$$\begin{aligned}
d\omega(\underline{A},\underline{B}) &= \underline{A}\big(\omega(\underline{B})\big) - \underline{B}\big(\omega(\underline{A})\big) - \omega\big([\underline{A},\underline{B}]\big) \\
&= \underline{A}(B) - \underline{B}(A) - \omega\big(\underline{[A,B]}\big) && \text{(Problem 27.1)} \\
&= 0 - 0 - [A,B].
\end{aligned}$$

Hence, (30.2) becomes

$$\Omega(X,Y) = -[A,B] + [A,B] = 0.$$

On the other hand,

$$(d\omega)_p(hX_p, hY_p) = (d\omega)_p(0,0) = 0.$$

(ii) Since the connection form ω is right-equivariant with respect to Ad,

$$\begin{aligned}
r_g^*\Omega &= r_g^*\left(d\omega + \frac{1}{2}[\omega,\omega]\right) && \text{(definition of curvature)}\\
&= dr_g^*\omega + \frac{1}{2}[r_g^*\omega, r_g^*\omega] && \text{(Proposition 21.8)}\\
&= d(\operatorname{Ad} g^{-1})\omega + \frac{1}{2}[(\operatorname{Ad} g^{-1})\omega, (\operatorname{Ad} g^{-1})\omega]\\
&= (\operatorname{Ad} g^{-1})\left(d\omega + \frac{1}{2}[\omega,\omega]\right)\\
&= (\operatorname{Ad} g^{-1})\Omega.
\end{aligned}$$

In this computation we used the fact that because $\operatorname{Ad} g^{-1} = (c_{g^{-1}})_*$ is the differential of a Lie group homomorphism, it is a Lie algebra homomorphism.

(iii) Taking the exterior derivative of the definition of the curvature form, we get

$$\begin{aligned}
d\Omega &= \frac{1}{2}d[\omega,\omega]\\
&= \frac{1}{2}([d\omega,\omega] - [\omega,d\omega]) && \text{(Proposition 21.6)}\\
&= [d\omega,\omega] && \text{(Proposition 21.5)}\\
&= \left[\Omega - \frac{1}{2}[\omega,\omega], \omega\right] && \text{(definition of } \Omega)\\
&= [\Omega,\omega] - \frac{1}{2}[[\omega,\omega],\omega]\\
&= [\Omega,\omega]. && \text{(Problem 21.5)}
\end{aligned}$$

□

In case P is the frame bundle $\operatorname{Fr}(E)$ of a rank r vector bundle E, with structure group $\operatorname{GL}(r,\mathbb{R})$, the second Bianchi identity becomes by Proposition 21.7

$$d\Omega = [\Omega,\omega] = \Omega\wedge\omega - \omega\wedge\Omega, \tag{30.3}$$

where the connection and curvature forms ω and Ω are $\mathfrak{gl}(r,\mathbb{R})$-valued forms on $\operatorname{Fr}(E)$. It should not be so surprising that it has the same form as the second Bianchi identity for the connection and curvature matrices relative to a frame e for E (Proposition 22.3). Indeed, by pulling back (30.3) by a frame $e\colon U \to \operatorname{Fr}(E)$, we get

$$\begin{aligned}
e^*d\Omega &= e^*(\Omega\wedge\omega) - e^*(\omega\wedge\Omega),\\
de^*\Omega &= (e^*\Omega)\wedge e^*\omega - (e^*\omega)\wedge e^*\Omega,\\
d\Omega_e &= \Omega_e\wedge\omega_e - \omega_e\wedge\Omega_e,
\end{aligned}$$

which is precisely Proposition 22.3.

Problems

30.1. Curvature of the Maurer–Cartan connection
Let G be a Lie group with Lie algebra $\mathfrak{g}$, and M a manifold. Compute the curvature of the Maurer–Cartan connection ω on the trivial bundle $\pi\colon M \times G \to M$.

30.2. Generalized second Bianchi identity on a frame bundle
Suppose $\mathrm{Fr}(E)$ is the frame bundle of a rank r vector bundle E over M. Let ω be an Ehresmann connection and Ω its curvature form on $\mathrm{Fr}(E)$. These are differential forms on $\mathrm{Fr}(E)$ with values in the Lie algebra $\mathfrak{gl}(r,\mathbb{R})$. Matrix multiplication and the Lie bracket on $\mathfrak{gl}(r,\mathbb{R})$ lead to two ways to multiply $\mathfrak{gl}(r,\mathbb{R})$-valued forms (see Section 21.5). We write Ω^k to denote the wedge product of Ω with itself k times. Prove that $d(\Omega^k) = [\Omega^k, \omega]$.

30.3. Lie bracket of horizontal vector fields
Let $P \to M$ be a principal bundle with a connection, and X, Y horizontal vector fields on P.

(a) Prove that $\Omega(X,Y) = -\omega([X,Y])$.
(b) Show that $[X,Y]$ is horizontal if and only if the curvature $\Omega(X,Y)$ equals zero.

§31 Covariant Derivative on a Principal Bundle

Throughout this chapter, G will be a Lie group with Lie algebra $\mathfrak{g}$ and V will be a finite-dimensional vector space. To a principal G-bundle $\pi\colon P \to M$ and a representation $\rho\colon G \to \mathrm{GL}(V)$, one can associate a vector bundle $P \times_\rho V \to M$ with fiber V. When ρ is the adjoint representation Ad of G on its Lie algebra $\mathfrak{g}$, the associated bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$ is called the ***adjoint bundle***, denoted by $\mathrm{Ad}\,P$.

Differential forms on M with values in the associated bundle $P \times_\rho V$ turn out to correspond in a one-to-one manner to certain V-valued forms on P called ***tensorial forms of type*** ρ. The curvature Ω of a connection ω on the principal bundle P is a $\mathfrak{g}$-valued tensorial 2-form of type Ad on P. Under this correspondence it may be viewed as a 2-form on M with values in the adjoint bundle $\mathrm{Ad}\,P$.

Using a connection ω, one can define a covariant derivative D of vector-valued forms on a principal bundle P. This covariant derivative maps tensorial forms to tensorial forms, and therefore induces a covariant derivative on forms on M with values in an associated bundle. In terms of the covariant derivative D, the curvature form is $\Omega = D\omega$, and Bianchi's second identity becomes $D\Omega = 0$.

31.1 The Associated Bundle

Let $\pi\colon P \to M$ be a principal G-bundle and $\rho\colon G \to \mathrm{GL}(V)$ a representation of G on a finite-dimensional vector space V. We write $\rho(g)v$ as $g \cdot v$ or even gv. The ***associated bundle*** $E := P \times_\rho V$ is the quotient of $P \times V$ by the equivalence relation

$$(p,v) \sim (pg, g^{-1} \cdot v) \qquad \text{for } g \in G \text{ and } (p,v) \in P \times V. \tag{31.1}$$

We denote the equivalence class of (p,v) by $[p,v]$. The associated bundle comes with a natural projection $\beta\colon P \times_\rho V \to M$, $\beta([p,v]) = \pi(p)$. Because

$$\beta([pg, g^{-1} \cdot v]) = \pi(pg) = \pi(p) = \beta([p,v]),$$

the projection β is well defined.

As a first example, the proposition below shows that an associated bundle of a trivial principal G-bundle is a trivial vector bundle.

Proposition 31.1. *If $\rho\colon G \to \mathrm{GL}(V)$ is a finite-dimensional representation of a Lie group G, and U is any manifold, then there is a fiber-preserving diffeomorphism*

$$\phi\colon (U \times G) \times_\rho V \xrightarrow{\sim} U \times V,$$
$$[(x,g),v] \mapsto (x, g \cdot v).$$

Proof. The map ϕ is well defined because if h is any element of G, then

$$\phi\big([(x,g)h, h^{-1} \cdot v]\big) = \big(x, (gh) \cdot h^{-1} \cdot v\big) = (x, g \cdot v) = \phi\big([(x,g),v]\big).$$

Define $\psi\colon U\times V\to (U\times G)\times_\rho V$ by

$$\psi(x,v) = [(x,1),v].$$

It is easy to check that ϕ and ψ are inverse to each other, are C^∞, and commute with the projections. □

Since a principal bundle $P\to M$ is locally $U\times G$, Proposition 31.1 shows that the associated bundle $P\times_\rho V\to M$ is locally trivial with fiber V. The vector space structure on V then makes $P\times_\rho V$ into a vector bundle over M:

$$\begin{aligned} [p,v_1]+[p,v_2] &= [p,v_1+v_2],\\ \lambda[p,v] &= [p,\lambda v], \quad \lambda\in\mathbb{R}. \end{aligned} \tag{31.2}$$

It is easy to show that these are well-defined operations not depending on the choice of $p\in E_x$ and that this makes the associated bundle $\beta\colon E\to M$ into a vector bundle (Problem 31.2).

Example 31.2. Let $\mathrm{Ad}\colon G\to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation of a Lie group G on its Lie algebra $\mathfrak{g}$. For a principal G-bundle $\pi\colon P\to M$, the associated vector bundle $\mathrm{Ad}\,P := P\times_{\mathrm{Ad}}\mathfrak{g}$ is called the ***adjoint bundle*** of P.

31.2 The Fiber of the Associated Bundle

If $\pi\colon P\to M$ is a principal G-bundle, $\rho\colon G\to\mathrm{GL}(V)$ is a representation, and $E := P\times_\rho V\to M$ is the associated bundle, we denote by P_x the fiber of P above $x\in M$, and by E_x the fiber of E above $x\in M$. For each $p\in P_x$, there is a canonical way of identifying the fiber E_x with the vector space V:

$$\begin{aligned} f_p\colon V &\to E_x,\\ v &\mapsto [p,v]. \end{aligned}$$

Lemma 31.3. *Let $\pi\colon P\to M$ be a principal G-bundle, $\rho\colon G\to\mathrm{GL}(V)$ a finite-dimensional representation, and $E=P\times_\rho V$ the associated vector bundle. For each point p in the fiber P_x, the map $f_p\colon V\to E_x$ is a linear isomorphism.*

Proof. Suppose $[p,v]=[p,w]$. Then $(p,w)=(pg,g^{-1}v)$ for some $g\in G$. Since G acts freely on P, the equality $p=pg$ implies that $g=1$. Hence, $w=g^{-1}v=v$. This proves that f_p is injective.

If $[q,w]$ is any point in E_x, then $q\in P_x$, so $q=pg$ for some $g\in G$. It follows that

$$[q,w]=[pg,w]=[p,gw]=f_p(gw).$$

This proves that f_p is surjective. □

The upshot is that every point p of the total space P of a principal bundle gives a linear isomorphism $f_p\colon V\to E_{\pi(p)}$ from V to the fiber of the associated bundle E above $\pi(p)$.

Lemma 31.4. *Let $E = P \times_\rho V$ be the vector bundle associated to the principal G-bundle $P \to M$ via the representation $\rho\colon G \to \mathrm{GL}(V)$, and $f_p\colon V \to E_x$ the linear isomorphism $v \mapsto [p, v]$. If $g \in G$, then $f_{pg} = f_p \circ \rho(g)$.*

Proof. For $v \in V$,

$$f_{pg}(v) = [pg, v] = [p, g \cdot v] = f_p(g \cdot v) = f_p\big(\rho(g)v\big). \qquad \square$$

Example 31.5. Let $\pi\colon P \to M$ be a principal G-bundle. The vector bundle $P \times_\rho V \to M$ associated to the trivial representation $\rho\colon G \to \mathrm{GL}(V)$ is the trivial bundle $M \times V \to M$, for there is a vector bundle isomorphism

$$P \times_\rho V \to M \times V,$$
$$[p, v] = [pg, g^{-1} \cdot v] = [pg, v] \mapsto \big(\pi(p), v\big),$$

with inverse map

$$(x, v) \mapsto [p, v] \quad \text{for any } p \in \pi^{-1}(x).$$

In this case, for each $p \in P$ the linear isomorphism $f_p\colon V \to E_x = V$, $v \mapsto [p, v]$, is the identity map.

31.3 Tensorial Forms on a Principal Bundle

We keep the same notation as in the previous section. Thus, $\pi\colon P \to M$ is a principal G-bundle, $\rho\colon G \to \mathrm{GL}(V)$ a finite-dimensional representation of G, and $E := P \times_\rho V$ the vector bundle associated to P via ρ.

Definition 31.6. A V-valued k-form φ on P is said to be ***right-equivariant of type*** ρ or ***right-equivariant with respect to*** ρ if for every $g \in G$,

$$r_g^* \varphi = \rho(g^{-1}) \cdot \varphi.$$

What this means is that for $p \in P$ and $v_1, \ldots, v_k \in T_pP$,

$$(r_g^* \varphi)_p(v_1, \ldots, v_k) = \rho(g^{-1})\big(\varphi_p(v_1, \ldots, v_k)\big).$$

In the literature (for example, [12, p. 75]), such a form is said to be ***pseudo-tensorial of type*** ρ.

Definition 31.7. A V-valued k-form φ on P is said to be ***horizontal*** if φ vanishes whenever one of its arguments is a vertical vector. Since a 0-form never takes an argument, every 0-form on P is by definition horizontal.

Definition 31.8. A V-valued k-form φ on P is ***tensorial of type*** ρ if it is right-equivariant of type ρ and horizontal. The set of all smooth tensorial V-valued k-forms of type ρ is denoted by $\Omega^k_\rho(P, V)$.

Example. Since the curvature Ω of a connection ω on a principal G-bundle P is horizontal and right-equivariant of type Ad, it is tensorial of type Ad.

The set $\Omega^k_\rho(P,V)$ of tensorial k-forms of type ρ on P becomes a vector space with the usual addition and scalar multiplication of forms. These forms are of special interest because they can be viewed as forms on the base manifold M with values in the associated bundle $E := P\times_\rho V$. To each tensorial V-valued k form $\varphi\in\Omega^k_\rho(P,V)$ we associate a k-form $\varphi^\flat\in\Omega^k(M,E)$ as follows. Given $x\in M$ and $v_1,\dots,v_k\in T_xM$, choose any point p in the fiber P_x and choose lifts $u_1,\dots,u_k$ at p of $v_1,\dots,v_k$, i.e., vectors in T_pP such that $\pi_*(u_i)=v_i$. Then $\varphi^\flat$ is defined by

$$\varphi^\flat_x(v_1,\dots,v_k) = f_p\bigl(\varphi_p(u_1,\dots,u_k)\bigr)\in E_x, \tag{31.3}$$

where $f_p\colon V\to E_x$ is the isomorphism $v\mapsto [p,v]$ of the preceding section.

Conversely, if $\psi\in\Omega^k(M,E)$, we define $\psi^\sharp\in\Omega^k_\rho(P,V)$ as follows. Given $p\in P$ and $u_1,\dots,u_k\in T_pP$, let $x=\pi(p)$ and set

$$\psi^\sharp_p(u_1,\dots,u_k) = f_p^{-1}\bigl(\psi_x(\pi_*u_1,\dots,\pi_*u_k)\bigr)\in V. \tag{31.4}$$

Theorem 31.9. *The map*

$$\begin{aligned}\Omega^k_\rho(P,V)&\to\Omega^k(M,E),\\ \varphi&\mapsto\varphi^\flat,\end{aligned}$$

is a well-defined linear isomorphism with inverse $\psi^\sharp \leftarrow\!\shortmid \psi$.

Proof. To show that $\varphi^\flat$ is well defined, we need to prove that the definition (31.3) is independent of the choice of $p\in P_x$ and of $u_1,\dots,u_k\in T_pP$. Suppose $u_1',\dots,u_k'\in T_pP$ is another set of vectors such that $\pi_*(u_i')=v_i$. Then $\pi_*(u_i'-u_i)=0$ so that $u_i'-u_i$ is vertical. Since φ is horizontal and k-linear,

$$\begin{aligned}\varphi_p(u_1',\dots,u_k') &= \varphi_p(u_1+\text{vertical},\dots,u_k+\text{vertical})\\ &= \varphi_p(u_1,\dots,u_k).\end{aligned}$$

This proves that for a given $p\in P$, the definition (31.3) is independent of the choice of lifts of $v_1,\dots,v_k$ to p.

Next suppose we choose pg instead of p as the point in the fiber P_x. Because $\pi\circ r_g=\pi$,

$$\pi_*(r_{g*}u_i) = (\pi\circ r_g)_*u_i = \pi_*u_i = v_i,$$

so that $r_{g*}u_1,\dots,r_{g*}u_k$ are lifts of $v_1,\dots,v_k$ to pg. We have, by right equivariance with respect to ρ,

$$\begin{aligned}\varphi_{pg}(r_{g*}u_1,\dots,r_{g*}u_k) &= (r_g^*\varphi_{pg})(u_1,\dots,u_k)\\ &= \rho(g^{-1})\varphi_p(u_1,\dots,u_k).\end{aligned}$$

So by Lemma 31.4,

$$\begin{aligned} f_{pg}\big(\varphi_{pg}(r_{g*}u_1,\dots,r_{g*}u_k)\big) &= f_{pg}\big(\rho(g^{-1})\varphi_p(u_1,\dots,u_k)\big) \\ &= \big(f_p\circ\rho(g)\big)\big(\rho(g^{-1})\varphi_p(u_1,\dots,u_k)\big) \\ &= f_p\big(\varphi_p(u_1,\dots,u_k)\big). \end{aligned}$$

This proves that the definition (31.3) is independent of the choice of p in the fiber P_x.

Let $\psi\in\Omega^k(M,E)$. It is clear from the definition (31.4) that $\psi^\sharp$ is horizontal. It is easy to show that $\psi^\sharp$ is right-equivariant with respect to ρ (Problem 31.4). Hence, $\psi^\sharp\in\Omega^k_\rho(P,V)$.

For $v_1,\dots,v_k\in T_xM$, choose $p\in P_x$ and vectors $u_1,\dots,u_k\in T_pP$ that lift $v_1,\dots,v_k$. Then

$$\begin{aligned} (\psi^{\sharp\flat})_x(v_1,\dots,v_k) &= f_p\big(\psi^\sharp_p(u_1,\dots,u_k)\big) \\ &= f_p\Big(f_p^{-1}\big(\psi_x(\pi_*u_1,\dots,\pi_*u_k)\big)\Big) \\ &= \psi_x(v_1,\dots,v_k). \end{aligned}$$

Hence, $\psi^{\sharp\flat}=\psi$.

Similarly, $\varphi^{\flat\sharp}=\varphi$ for $\varphi\in\Omega^k_\rho(P,V)$, which we leave to the reader to show (Problem 31.5). Therefore, the map $\psi\mapsto\psi^\sharp$ is inverse to the map $\varphi\mapsto\varphi^\flat$. □

Example 31.10 (*Curvature as a form on the base*). By Theorem 31.9, the curvature form Ω of a connection on a principal G-bundle P can be viewed as an element of $\Omega^2(M,\mathrm{Ad}\,P)$, a 2-form on M with values in the adjoint bundle $\mathrm{Ad}\,P$.

When $k=0$ in Theorem 31.9, $\Omega^0_p(P,V)$ consists of maps $f\colon P\to V$ that are right-equivariant with respect to ρ:

$$(r_g^*f)(p)=\rho(g)^{-1}f(p),$$

or

$$f(pg)=\rho(g^{-1})f(p)=g^{-1}\cdot f(p).$$

On the right-hand side of Theorem 31.9,

$$\Omega^0(M,P\times_\rho V)=\Omega^0(M,E)=\text{ sections of the associated bundle } E.$$

Hence, we have the following corollary.

Corollary 31.11. *Let G be a Lie group, $P\to M$ a principal G-bundle, and $\rho\colon G\to\mathrm{Aut}(V)$ a representation of G. There is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{G-equivariant maps}\\ f\colon P\to V\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{sections of the associated bundle}\\ P\times_\rho V\to M\end{matrix}\right\}.$$

By the local triviality condition, for any principal bundle $\pi\colon P\to M$ the projection map π is a submersion and therefore the pullback map $\pi^*\colon\Omega^*(M)\to\Omega^*(P)$ is an injection. A differential form φ on P is said to be ***basic*** if it is the pullback $\pi^*\psi$

of a form ψ on M; it is ***G-invariant*** if $r_g^*\varphi = \varphi$ for all $g \in G$. More generally, for any vector space V, these concepts apply to V-valued forms as well.

Suppose $\rho\colon G \to \mathrm{GL}(V)$ is the trivial representation $\rho(g) = \mathbb{1}$ for all $g \in G$. Then an equivariant form φ of type ρ on P satisfies

$$r_g^*\varphi = \rho(g^{-1}) \cdot \varphi = \varphi \quad \text{for all } g \in G.$$

Thus, an equivariant form of type ρ for the trivial representation ρ is exactly an invariant form on P. Unravelling Theorem 31.9 for a trivial representation will give the following theorem.

Theorem 31.12. *Let $\pi\colon P \to M$ be a principal G-bundle and V a vector space. A V-valued form on P is basic if and only if it is horizontal and G-invariant.*

Proof. Let $\rho\colon G \to \mathrm{GL}(V)$ be the trivial representation. As noted above, $\Omega_\rho^k(P,V)$ consists of horizontal, G-invariant V-valued k-forms on P.

By Example 31.5, when ρ is the trivial representation, the vector bundle $E = P \times_\rho V$ is the product bundle $M \times V$ over M and for each $p \in P$, the linear isomorphism $f_p\colon V \to E_x = V$, where $x = \pi(p)$, is the identity map. Then the isomorphism

$$\Omega^k(M,E) = \Omega^k(M, M \times V) = \Omega^k(M,V) \to \Omega_\rho^k(P,V),$$
$$\psi \mapsto \psi^\#,$$

is given by

$$\begin{aligned}\psi_p^\#(u_1,\ldots,u_k) &= \psi_x(\pi_*u_1,\ldots,\pi_*u_k) \quad \text{(by (31.4))}\\ &= (\pi^*\psi)_p(u_1,\ldots,u_k).\end{aligned}$$

Therefore,

$$\psi^\# = \pi^*\psi.$$

This proves that horizontal, G-invariant forms on P are precisely the basic forms. □

31.4 Covariant Derivative

Recall that the existence of a connection ω on a principal G-bundle $\pi\colon P \to M$ is equivalent to the decomposition of the tangent bundle TP into a direct sum of the vertical subbundle $\mathcal{V}$ and a smooth right-invariant horizontal subbundle $\mathcal{H}$. For any vector $X_p \in T_pP$, we write

$$X_p = vX_p + hX_p$$

as the sum of its vertical and horizontal components. This will allow us to define a covariant derivative of vector-valued forms on P. By the isomorphism of Theorem 31.9, we obtain in turn a covariant derivative of forms on M with values in an associated bundle.

Let $\rho\colon G \to \mathrm{GL}(V)$ be a finite-dimensional representation of G and let $E := P \times_\rho V$ be the associated vector bundle.

Proposition 31.13. *If $\varphi \in \Omega^k(P,V)$ is right-equivariant of type ρ, then so is $d\varphi$.*

Proof. For a fixed $g \in G$,

$$\begin{aligned} r_g^* d\varphi &= d r_g^* \varphi = d\rho(g^{-1})\varphi \\ &= \rho(g^{-1}) d\varphi, \end{aligned}$$

since $\rho(g^{-1})$ is a constant linear map for a fixed g. □

In general, the exterior derivative does not preserve horizontality. For any V-valued k-form φ on P, we define its ***horizontal component*** $\varphi^h \in \Omega^k(P,V)$ as follows: for $p \in P$ and $v_1, \ldots, v_k \in T_pP$,

$$\varphi_p^h(v_1, \ldots, v_k) = \varphi_p(hv_1, \ldots, hv_k).$$

Proposition 31.14. *If $\varphi \in \Omega^k(P,V)$ is right-equivariant of type ρ, then so is φ^h.*

Proof. For $g \in G$, $p \in P$, and $v_1, \ldots, v_k \in T_pP$,

$$\begin{aligned} r_g^*(\varphi_{pg}^h)(v_1, \ldots, v_k) &= \varphi_{pg}^h(r_{g*}v_1, \ldots, r_{g*}v_k) && \text{(definition of pullback)} \\ &= \varphi_{pg}(hr_{g*}v_1, \ldots, hr_{g*}v_k) && \text{(definition of } \varphi^h) \\ &= \varphi_{pg}(r_{g*}hv_1, \ldots, r_{g*}hv_k) && \text{(Proposition 28.4)} \\ &= (r_g^*\varphi_{pg})(hv_1, \ldots, hv_k) \\ &= \rho(g^{-1}) \cdot \varphi_p(hv_1, \ldots, hv_k) && \text{(right-equivariance of } \varphi) \\ &= \rho(g^{-1}) \cdot \varphi_p^h(v_1, \ldots, v_k) \end{aligned}$$ □

Propositions 31.13 and 31.14 together imply that if $\varphi \in \Omega^k(P,V)$ is right-equivariant of type ρ, then $(d\varphi)^h \in \Omega^{k+1}(P,V)$ is horizontal and right-equivariant of type ρ, i.e., tensorial of type ρ.

Definition 31.15. Let $\pi\colon P \to M$ be a principal G-bundle with a connection ω and let V be a real vector space. The ***covariant derivative*** of a V-valued k-form $\varphi \in \Omega^k(P,V)$ is $D\varphi = (d\varphi)^h$.

Let $\rho\colon G \to \mathrm{GL}(V)$ be a finite-dimensional representation of the Lie group G. The covariant derivative is defined for any V-valued k-form on P, and it maps a right-equivariant form of type ρ to a tensorial form of type ρ. In particular, it restricts to a map

$$D\colon \Omega_\rho^k(P,V) \to \Omega_\rho^{k+1}(P,V) \tag{31.5}$$

on the space of tensorial forms.

Proposition 31.16. *Let $\pi\colon P \to M$ be a principal G-bundle with a connection and $\rho\colon G \to \mathrm{GL}(V)$ a representation of G. The covariant derivative*

$$D\colon \Omega_\rho^k(P,V) \to \Omega_\rho^{k+1}(P,V)$$

on tensorial forms of type ρ is an antiderivation of degree $+1$.

Proof. Let $\omega, \tau \in \Omega^*_\rho(P,V)$ be tensorial forms of type ρ. Then

$$\begin{aligned} D(\omega\wedge\tau) &= \big(d(\omega\wedge\tau)\big)^h \\ &= \big((d\omega)\wedge\tau + (-1)^{\deg\omega}\omega\wedge d\tau\big)^h \\ &= (d\omega)^h\wedge\tau^h + (-1)^{\deg\omega}\omega^h\wedge(d\tau)^h \\ &= Dw\wedge\tau^h + (-1)^{\deg\omega}\omega^h\wedge D\tau. \end{aligned}$$

Since τ and ω are horizontal, $\tau^h = \tau$ and $\omega^h = \omega$. Therefore,

$$D(\omega\wedge\tau) = D\omega\wedge\tau + (-1)^{\deg\omega}\omega\wedge D\tau. \qquad \square$$

If $E := P\times_\rho V$ is the associated vector bundle via the representation ρ, then the isomorphism of Theorem 31.9 transforms the linear map (31.5) into a linear map

$$D\colon \Omega^k(M,E)\to\Omega^{k+1}(M,E).$$

Unlike the exterior derivative, the covariant derivative depends on the choice of a connection on P. Moreover, $D^2 \neq 0$ in general.

Example 31.17 (*Curvature of a principal bundle*). By Theorem 30.4 the curvature form $\Omega \in \Omega^2_{\mathrm{Ad}}(P,\mathfrak{g})$ on a principal bundle is the covariant derivative $D\omega$ of the connection form $\omega \in \Omega^1(P,\mathfrak{g})$. Because ω is not horizontal, it is not in $\Omega^1_{\mathrm{Ad}}(P,\mathfrak{g})$.

31.5 A Formula for the Covariant Derivative of a Tensorial Form

Let $\pi\colon P\to M$ be a smooth principal G-bundle with a connection ω, and let $\rho\colon G\to \mathrm{GL}(V)$ be a finite-dimensional representation of G. In the preceding section we defined the covariant derivative of a V-valued k-form φ on P: $D\varphi = (d\varphi)^h$, the horizontal component of $d\varphi$. In this section we derive a useful alternative formula for the covariant derivative, but only for a tensorial form.

The Lie group representation $\rho\colon G\to\mathrm{GL}(V)$ induces a Lie algebra representation $\rho_*\colon \mathfrak{g}\to\mathfrak{gl}(V)$, which allows us to define a product of a $\mathfrak{g}$-valued k-form τ and a V-valued ℓ-form φ on P: for $p\in P$ and $v_1,\dots,v_{k+\ell}\in T_pP$,

$$\begin{aligned} &(\tau\cdot\varphi)_p(v_1,\dots,v_{k+\ell}) \\ &\qquad = \frac{1}{k!\ell!}\sum_{\sigma\in S_{k+\ell}} \mathrm{sgn}(\sigma)\rho_*\big(\tau_p(v_{\sigma(1)},\dots,v_{\sigma(k)})\big)\varphi_p\left(v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)}\right). \end{aligned}$$

For the same reason as the wedge product, $\tau\cdot\varphi$ is multilinear and alternating in its arguments; it is therefore a $(k+\ell)$-covector with values in V.

Example 31.18. If $V=\mathfrak{g}$ and $\rho = \mathrm{Ad}\colon G\to\mathrm{GL}(\mathfrak{g})$ is the adjoint representation, then

$$(\tau\cdot\varphi)_p = \frac{1}{k!\ell!}\sum_{\sigma\in S_{k+\ell}}\mathrm{sgn}(\sigma)\left[\tau_p(v_{\sigma(1)},\dots,v_{\sigma(k)}),\varphi_p(v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)})\right].$$

In this case we also write $[\tau,\varphi]$ instead of $\tau\cdot\varphi$.

Theorem 31.19. *Let $\pi\colon P \to M$ be a principal G-bundle with connection form ω, and $\rho\colon G \to \mathrm{GL}(V)$ a finite-dimensional representation of G. If $\varphi \in \Omega^k_\rho(P,V)$ is a V-valued tensorial form of type ρ, then its covariant derivative is given by*

$$D\varphi = d\varphi + \omega \cdot \varphi.$$

Proof. Fix $p \in P$ and $v_1,\dots,v_{k+1} \in T_pP$. We need to show that

$$(d\varphi)_p(hv_1,\dots,hv_{k+1}) = (d\varphi)_p(v_1,\dots,v_{k+1}) + \frac{1}{k!}\sum_{\sigma\in S_{k+1}} \operatorname{sgn}(\sigma)\rho_*\left(\omega_p(v_{\sigma(1)})\right)\varphi_p\left(v_{\sigma(2)},\dots,v_{\sigma(k+1)}\right). \tag{31.6}$$

Because both sides of (31.6) are linear in each argument v_i, which may be decomposed into the sum of a vertical and a horizontal component, we may assume that each v_i is either vertical or horizontal. By Lemma 30.3, throughout the proof we may further assume that the vectors $v_1,\dots,v_{k+1}$ have been extended to vector fields $X_1,\dots,X_{k+1}$ on P each of which is either vertical or horizontal. If X_i is vertical, then it is a fundamental vector field $\underline{A_i}$ for some $A_i \in \mathfrak{g}$. If X_i is horizontal, then it is the horizontal lift $\tilde{B}_i$ of a vector field B_i on M. By construction, $\tilde{B}_i$ is right-invariant (Proposition 28.6).

Instead of proving (31.6) at a point p, we will prove the equality of functions

$$(d\varphi)(hX_1,\dots,hX_{k+1}) = \mathrm{I} + \mathrm{II}, \tag{31.7}$$

where

$$\mathrm{I} = (d\varphi)(X_1,\dots,X_{k+1})$$

and

$$\mathrm{II} = \frac{1}{k!}\sum_{\sigma\in S_{k+1}} \operatorname{sgn}(\sigma)\rho_*\left(\omega(X_{\sigma(1)})\right)\varphi\left(X_{\sigma(2)},\dots,X_{\sigma(k+1)}\right).$$

Case 1. *The vector fields $X_1,\dots,X_{k+1}$ are all horizontal.*
Then $\mathrm{II} = 0$ because $\omega(X_{\sigma(1)}) = 0$ for all $\sigma \in S_{k+1}$. In this case, (31.7) is trivially true.

Case 2. *At least two of $X_1,\dots,X_{k+1}$ are vertical.*
By the skew-symmetry of the arguments, we may assume that $X_1 = \underline{A_1}$ and $X_2 = \underline{A_2}$ are vertical. By Problem 27.1, $[X_1,X_2] = \underline{[A_1,A_2]}$ is also vertical.

The left-hand side of (31.7) is zero because $hX_1 = 0$. By the global formula for the exterior derivative [21, Th. 20.14, p. 233],

$$\mathrm{I} = \sum_{i=1}^{k+1}(-1)^{i-1}X_i\varphi(\dots,\widehat{X_i},\dots) + \sum_{1\le i<j\le k+1}(-1)^{i+j}\varphi([X_i,X_j],\dots,\widehat{X_i},\dots,\widehat{X_j},\dots).$$

In this expression every term in the first sum is zero because φ is horizontal and at least one of its arguments is vertical. In the second sum at least one of the arguments of φ is X_1, X_2, or $[X_1,X_2]$, all of which are vertical. Therefore, every term in the second sum in I is also zero.

As for II in (31.7), in every term at least one of the arguments of φ is vertical, so $\mathrm{II} = 0$.

Case 3. *The first vector field $X_1 = \underline{A}$ is vertical; the rest $X_2, \ldots, X_{k+1}$ are horizontal and right-invariant.*
The left-hand side of (31.7) is clearly zero because $hX_1 = 0$.

On the right-hand side,

$$\begin{aligned}\mathrm{I} &= (d\varphi)(X_1,\ldots,X_{k+1}) \\ &= \sum(-1)^{i+1}X_i\varphi(X_1,\ldots,\widehat{X_i},\ldots,X_{k+1}) \\ &\qquad\qquad + \sum(-1)^{i+j}\varphi([X_i,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_{k+1}).\end{aligned}$$

Because φ is horizontal and X_1 is vertical, the only nonzero term in the first sum is

$$X_1\varphi(X_2,\ldots,X_{k+1}) = \underline{A}\varphi(X_2,\ldots,X_{k+1})$$

and the only nonzero terms in the second sum are

$$\sum_{j=2}^{k+1}(-1)^{1+j}\varphi([X_1,X_j],\widehat{X_1},X_2,\ldots,\widehat{X_j},\ldots,X_{k+1}).$$

Since the X_j, $j = 2,\ldots,k+1$, are right-invariant horizontal vector fields, by Lemma 28.7,

$$[X_1,X_j] = [\underline{A},X_j] = 0.$$

Therefore,

$$\mathrm{I} = \underline{A}\varphi(X_2,\ldots,X_{k+1}).$$

If $\sigma(i) = 1$ for any $i \geq 2$, then

$$\varphi(X_{\sigma(2)},\ldots,X_{\sigma(k+1)}) = 0.$$

It follows that the nonzero terms in II all satisfy $\sigma(1) = 1$ and

$$\begin{aligned}\mathrm{II} &= \frac{1}{k!}\sum_{\substack{\sigma\in S_{k+1}\\ \sigma(1)=1}}\operatorname{sgn}(\sigma)\rho_*\big(\omega(X_1)\big)\varphi\left(X_{\sigma(2)},\ldots,X_{\sigma(k+1)}\right) \\ &= \frac{1}{k!}\sum_{\substack{\sigma\in S_{k+1}\\ \sigma(1)=1}}\operatorname{sgn}(\sigma)\rho_*(A)\varphi\left(X_{\sigma(2)},\ldots,X_{\sigma(k+1)}\right) \\ &= \rho_*(A)\varphi\left(X_2,\ldots,X_{k+1}\right) \qquad \text{(because } \varphi \text{ is alternating).}\end{aligned}$$

Denote by f the function $\varphi(X_2,\dots,X_{k+1})$ on P. For $p \in P$, to calculate $\underline{A}_p f$, choose a curve $c(t)$ in G with initial point $c(0) = e$ and initial vector $c'(0) = A$, for example, $c(t) = \exp(tA)$. Then with $j_p\colon G \to P$ being the map $j_p(g) = p \cdot g$,

$$\begin{aligned}\underline{A}_p f &= j_{p*}(A) f = j_{p*}\big(c'(0)\big) f = j_{p*}\left(c_*\left(\frac{d}{dt}\bigg|_{t=0}\right)\right) f \\ &= (j_p \circ c)_*\left(\frac{d}{dt}\bigg|_{t=0}\right) f = \frac{d}{dt}\bigg|_{t=0} (f \circ j_p \circ c).\end{aligned}$$

By the right-invariance of the horizontal vector fields $X_2,\dots,X_{k+1}$,

$$\begin{aligned}(f \circ j_p \circ c)(t) &= f\big(pc(t)\big) \\ &= \varphi_{pc(t)}\left(X_{2,pc(t)},\dots,X_{k+1,pc(t)}\right) \\ &= \varphi_{pc(t)}\left(r_{c(t)*}X_{2,p},\dots,r_{c(t)*}X_{k+1,p}\right) \\ &= r_{c(t)}^*\varphi_{pc(t)}\left(X_{2,p},\dots,X_{k+1,p}\right) \\ &= \rho(c(t)^{-1})\varphi_p\left(X_{2,p},\dots,X_{k+1,p}\right) \qquad \text{(right-equivariance of } \varphi) \\ &= \rho(c(t)^{-1}) f(p).\end{aligned}$$

Differentiating this expression with respect to t and using the fact that the differential of the inverse is the negative [21, Problem 8.8(b)], we have

$$\underline{A}_p f = (f \circ j_p \circ c)'(0) = -\rho_*(c'(0)) f(p) = -\rho_*(A) f(p).$$

So the right-hand side of (31.7) is

$$\mathrm{I} + \mathrm{II} = \underline{A} f + \rho_*(A) f = -\rho_*(A) f + \rho_*(A) f = 0. \qquad \square$$

If V is the Lie algebra $\mathfrak{g}$ of a Lie group G and ρ is the adjoint representation of G, then $\omega \cdot \varphi = [\omega, \varphi]$. In this case, for any tensorial k-form $\varphi \in \Omega^k_{\mathrm{Ad}}(P,\mathfrak{g})$,

$$D\varphi = d\varphi + [\omega, \varphi].$$

Although the covariant derivative is defined for any V-valued form on P, Theorem 31.19 is true only for tensorial forms. Since the connection form ω is not tensorial, Theorem 31.19 cannot be applied to ω. In fact, by the definition of the curvature form,

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega].$$

By Theorem 30.4, $\Omega = (d\omega)^h = D\omega$. Combining these two expressions for the curvature, one obtains

$$D\omega = d\omega + \frac{1}{2}[\omega, \omega].$$

The factor of $1/2$ shows that Theorem 31.19 is not true when applied to ω.

Since the curvature form Ω on a principal bundle P is tensorial of type Ad, Theorem 31.19 applies and the second Bianchi identity (Theorem 30.4) may be restated as

$$D\Omega = d\Omega + [\omega, \Omega] = 0. \tag{31.8}$$

Problems

Unless otherwise specified, in the following problems G is a Lie group with Lie algebra $\mathfrak{g}$, $\pi\colon P \to M$ a principal G-bundle, $\rho\colon G \to \mathrm{GL}(V)$ a finite-dimensional representation of G, and $E = P \times_\rho V$ the associated bundle.

31.1. Transition functions of an associated bundle
Show that if $\{(U_\alpha, \phi_\alpha)\}$ is a trivialization for P with transition functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$, then there is a trivialization $\{(U_\alpha, \psi_\alpha)\}$ for E with transition functions $\rho \circ g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(V)$.

31.2. Vector bundle structure on an associated bundle
Show that the operations (31.2) on $E = P \times_\rho V$ are well defined and make the associated bundle $\beta\colon E \to M$ into a vector bundle.

31.3. Associated bundle of a frame bundle
Let $E \to M$ be a vector bundle of rank r and $\mathrm{Fr}(E) \to M$ its frame bundle. Show that the vector bundle associated to $\mathrm{Fr}(E)$ via the identity representation $\rho\colon \mathrm{GL}(r,\mathbb{R}) \to \mathrm{GL}(r,\mathbb{R})$ is isomorphic to E.

31.4. Tensorial forms
Prove that if $\psi \in \Omega^k(M, P \times_\rho V)$, then $\psi^\sharp \in \Omega^k(P,V)$ is right-equivariant with respect to ρ.

31.5. Tensorial forms
For $\varphi \in \Omega^k_\rho(P,V)$, prove that $\varphi^{\flat\sharp} = \varphi$.

§32 Characteristic Classes of Principal Bundles

To a real vector bundle $E \to M$ of rank r, one can associate its frame bundle $\mathrm{Fr}(E) \to M$, a principal $\mathrm{GL}(r,\mathbb{R})$-bundle. Similarly, to a complex vector bundle of rank r, one can associate its frame bundle, a principal $\mathrm{GL}(r,\mathbb{C})$-bundle and to an oriented real vector bundle of rank r, one can associate its oriented frame bundle, a principal $\mathrm{GL}^+(r,\mathbb{R})$-bundle, where $\mathrm{GL}^+(r,\mathbb{R})$ is the group of all $r \times r$ matrices of positive determinant. The Pontrjagin classes of a real vector bundle, the Chern classes of a complex vector bundle, and the Euler class of an oriented real vector bundle may be viewed as characteristic classes of the associated principal G-bundle for $G = \mathrm{GL}(r,\mathbb{R}), \mathrm{GL}(r,\mathbb{C})$, and $\mathrm{GL}^+(r,\mathbb{R})$, respectively.

In this section we will generalize the construction of characteristic classes to principal G-bundles for any Lie group G. These are some of the most important diffeomorphism invariants of a principal bundle.

32.1 Invariant Polynomials on a Lie Algebra

Let V be a vector space of dimension n and $V^\vee$ its dual space. An element of $\mathrm{Sym}^k(V^\vee)$ is called a ***polynomial*** of degree k on V. Relative to a basis $e_1, \dots, e_n$ for V and corresponding dual basis $\alpha^1, \dots, \alpha^n$ for $V^\vee$, a function $f\colon V \to \mathbb{R}$ is a polynomial of degree k if and only if it is expressible as a sum of monomials of degree k in $\alpha^1, \dots, \alpha^n$:

$$f = \sum a_I \alpha^{i_1} \cdots \alpha^{i_k}. \tag{32.1}$$

For example, if $V = \mathbb{R}^{n\times n}$ is the vector space of all $n \times n$ matrices, then $\mathrm{tr} X$ is a polynomial of degree 1 on V and $\det X$ is a polynomial of degree n on V.

Suppose now that $\mathfrak{g}$ is the Lie algebra of a Lie group G. A polynomial $f\colon \mathfrak{g} \to \mathbb{R}$ is said to be $\mathrm{Ad}(G)$-***invariant*** if for all $g \in G$ and $X \in \mathfrak{g}$,

$$f\big((\mathrm{Ad}\, g)X\big) = f(X).$$

For example, if G is the general linear group $\mathrm{GL}(n,\mathbb{R})$, then $(\mathrm{Ad}\, g)X = gXg^{-1}$ and $\mathrm{tr} X$ and $\det X$ are $\mathrm{Ad}\, G$-invariant polynomials on the Lie algebra $\mathfrak{gl}(n,\mathbb{R})$.

32.2 The Chern–Weil Homomorphism

Let G be a Lie group with Lie algebra $\mathfrak{g}$, $P \to M$ a principal G-bundle, ω an Ehresmann connection on P, and Ω the curvature form of ω. Fix a basis $e_1, \dots, e_n$ for $\mathfrak{g}$ and dual basis $\alpha^1, \dots, \alpha^n$ for $\mathfrak{g}^\vee$. Then the curvature form Ω is a linear combination

$$\Omega = \sum \Omega^i e_i,$$

where the coefficients Ω^i are real-valued 2-forms on P. If $f\colon \mathfrak{g} \to \mathbb{R}$ is the polynomial $\sum a_I \alpha^{i_1} \cdots \alpha^{i_k}$, we define $f(\Omega)$ to be the $2k$-form

$$f(\Omega) = \sum a_I \Omega^{i_1} \wedge \cdots \wedge \Omega^{i_k}$$

on P. Although defined in terms of a basis for $\mathfrak{g}$, the $2k$-form $f(\Omega)$ is independent of the choice of a basis (Problem 32.2).

Recall that the ***covariant derivative*** $D\varphi$ of a k-form φ on a principal bundle P is given by

$$(D\varphi)_p(v_1,\dots,v_k) = (d\varphi)_p(hv_1,\dots,hv_k),$$

where $v_i \in T_pP$ and hv_i is the horizontal component of v_i.

Lemma 32.1. *Let $\pi\colon P \to M$ be a principal bundle. If φ is a basic form on P, then $d\varphi = D\varphi$.*

Proof. A tangent vector $X_p \in T_pP$ decomposes into the sum of its vertical and horizontal components:

$$X_p = vX_p + hX_p.$$

Here $h\colon T_pP \to T_pP$ is the map that takes a tangent vector to its horizontal component. Since $\pi_* X_p = \pi_* hX_p$ for all $X_p \in T_pP$, we have

$$\pi_* = \pi_* \circ h.$$

Suppose $\varphi = \pi^*\tau$ for $\tau \in \Omega^k(M)$. Then

$$\begin{aligned}
D\varphi &= (d\varphi)\circ h && \text{(definition of } D)\\
&= (d\pi^*\tau)\circ h && (\varphi \text{ is basic})\\
&= (\pi^* d\tau)\circ h && \text{([21, Prop. 19.5])}\\
&= d\tau \circ \pi_* \circ h && \text{(definition of } \pi^*)\\
&= d\tau \circ \pi_* && (\pi_* \circ h = \pi_*)\\
&= \pi^* d\tau && \text{(definition of } \pi^*)\\
&= d\pi^*\tau && \text{([21, Prop. 19.5])}\\
&= d\varphi && (\varphi = \pi^*\tau).
\end{aligned}$$

□

The Chern–Weil homomorphism is based on the following theorem. As before, G is a Lie group with Lie algebra $\mathfrak{g}$.

Theorem 32.2. *Let Ω be the curvature of a connection ω on a principal G-bundle $\pi\colon P \to M$, and f an $\mathrm{Ad}(G)$-invariant polynomial of degree k on $\mathfrak{g}$. Then*

(i) *$f(\Omega)$ is a basic form on P, i.e., there exists a $2k$-form Λ on M such that $f(\Omega) = \pi^*\Lambda$.*
(ii) *Λ is a closed form.*
(iii) *The cohomology class $[\Lambda]$ is independent of the connection.*

Proof. (i) Since the curvature Ω is horizontal, so are its components Ω^i and therefore so is $f(\Omega) = \sum a_I \Omega^{i_1} \wedge \cdots \wedge \Omega^{i_k}$.

To check the G-invariance of $f(\Omega)$, let $g \in G$. Then

$$\begin{aligned} r_g^*\big(f(\Omega)\big) &= r_g^*\big(\sum a_I \Omega^{i_1} \wedge \cdots \wedge \Omega^{i_k}\big) \\ &= \sum a_I r_g^*(\Omega^{i_1}) \wedge \cdots \wedge r_g^*(\Omega^{i_k}). \end{aligned}$$

Since the curvature form Ω is right-equivariant,

$$r_g^*\Omega = (\operatorname{Ad} g^{-1})\Omega$$

or

$$r_g^*(\sum \Omega^i e_i) = \sum \big((\operatorname{Ad} g^{-1})\Omega\big)^i e_i,$$

so that

$$r_g^*(\Omega^i) = \big((\operatorname{Ad} g^{-1})\Omega\big)^i.$$

Thus,

$$\begin{aligned} r_g^*\big(f(\Omega)\big) &= \sum a_I \big((\operatorname{Ad} g^{-1})\Omega\big)^{i_1} \wedge \cdots \wedge \big((\operatorname{Ad} g^{-1})\Omega\big)^{i_k} \\ &= f\big((\operatorname{Ad} g^{-1})\Omega\big) \\ &= f(\Omega) \qquad \text{(by the } \operatorname{Ad} G\text{-invariance of } f). \end{aligned}$$

Since $f(\Omega)$ is horizontal and G-invariant, by Theorem 31.12, it is basic.

(ii) Since $\pi_* \colon T_pP \to T_{\pi(p)}M$ is surjective, $\pi^* \colon \Omega^*(M) \to \Omega^*(P)$ is injective. Therefore, to show that $d\Lambda = 0$, it suffices to show that

$$\pi^* d\Lambda = d\pi^*\Lambda = df(\Omega) = 0.$$

If $f = \sum a_I \alpha^{i_1} \cdots \alpha^{i_k}$, then

$$f(\Omega) = \sum a_I \Omega^{i_1} \wedge \cdots \wedge \Omega^{i_k}.$$

In this expression, each a_I is a constant and therefore by Lemma 32.1

$$Da_I = da_I = 0.$$

By the second Bianchi identity (31.8), $D\Omega = 0$. Therefore, $D\Omega^i = 0$ for each i. Since the Ω^i are right-equivariant of type Ad and horizontal, they are tensorial forms. By Lemma 32.1 and because D is an antiderivation on tensorial forms (Proposition 31.16)

$$\begin{aligned} d\big(f(\Omega)\big) &= D\big(f(\Omega)\big) = D\big(\sum a_I \Omega^{i_1} \wedge \cdots \wedge \Omega^{i_k}\big) \\ &= \sum_I \sum_j a_I \Omega^{i_1} \wedge \cdots \wedge D\Omega^{i_j} \wedge \cdots \wedge \Omega^{i_{2k}} \\ &= 0. \end{aligned}$$

(iii) Let I be an open interval containing the closed interval $[0,1]$. Then $P \times I$ is a principal G-bundle over $M \times I$. Denote by ρ the projection $P \times I \to P$ to the first factor. If ω_0 and ω_1 are two connections on P, then

$$\tilde{\omega} = (1-t)\rho^*\omega_0 + t\rho^*\omega_1 \tag{32.2}$$

is a connection on $P \times I$ (Check the details). Moreover, if $i_t \colon P \to P \times I$ is the inclusion $p \mapsto (p,t)$, then $i_0^*\tilde{\omega} = \omega_0$ and $i_1^*\tilde{\omega} = \omega_1$.

Let

$$\tilde{\Omega} = d\tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}]$$

be the curvature of the connection $\tilde{\omega}$. It pulls back under i_0 to

$$\begin{aligned} i_0^*\tilde{\Omega} &= di_0^*\tilde{\omega} + \frac{1}{2} i_0^*[\tilde{\omega}, \tilde{\omega}] \\ &= d\omega_0 + \frac{1}{2}[i_0^*\tilde{\omega}, i_0^*\tilde{\omega}] \\ &= d\omega_0 + \frac{1}{2}[\omega_0, \omega_0] \\ &= \Omega_0, \end{aligned}$$

the curvature of the connection ω_0. Similarly, $i_1^*\tilde{\Omega} = \Omega_1$, the curvature of the connection ω_1.

For any $\mathrm{Ad}(G)$-invariant polynomial

$$f = \sum a_I \alpha^{i_1} \cdots \alpha^{i_k}$$

of degree k on $\mathfrak{g}$,

$$\begin{aligned} i_0^* f(\tilde{\Omega}) &= i_0^* \sum a_I \tilde{\Omega}^{i_1} \wedge \cdots \wedge \tilde{\Omega}^{i_k} \\ &= \sum a_I \Omega_0^{i_1} \wedge \cdots \wedge \Omega_0^{i_k} \\ &= f(\Omega_0) \end{aligned}$$

and

$$i_1^* f(\tilde{\Omega}) = f(\Omega_1).$$

Note that i_0 and $i_1 \colon P \to P \times I$ are homotopic through the homotopy i_t. By the homotopy axiom of de Rham cohomology, the cohomology classes $[i_0^* f(\tilde{\Omega})]$ and $[i_1^* f(\tilde{\Omega})]$ are equal. Thus, $[f(\Omega_0)] = [f(\Omega_1)]$, or

$$\pi^*[\Lambda_0] = \pi^*[\Lambda_1].$$

By the injectivity of π^*, $[\Lambda_0] = [\Lambda_1]$, so the cohomology class of Λ is independent of the connection. □

Let $\pi \colon P \to M$ be a principal G-bundle with curvature form Ω. To every $\mathrm{Ad}(G)$-invariant polynomial on $\mathfrak{g}$, one can associate the cohomology class $[\Lambda] \in H^*(M)$ such that $f(\Omega) = \pi^*\Lambda$. The cohomology class $[\Lambda]$ is called the ***characteristic class*** of P associated to f. Denote by $\mathrm{Inv}(\mathfrak{g})$ the algebra of all $\mathrm{Ad}(G)$-invariant polynomials on $\mathfrak{g}$. The map

$$\begin{aligned} w \colon \mathrm{Inv}(\mathfrak{g}) &\to H^*(M) \\ f &\mapsto [\Lambda], \text{ where } f(\Omega) = \pi^*\Lambda, \end{aligned} \tag{32.3}$$

that maps each $\mathrm{Ad}(G)$-invariant polynomial to its characteristic class is called the ***Chern–Weil homomorphism***.

Example 32.3. If the Lie group G is $\mathrm{GL}(r,\mathbb{C})$, then by Theorem B.10 the ring of $\mathrm{Ad}(G)$-invariant polynomials on $\mathfrak{gl}(r,\mathbb{C})$ is generated by the coefficients $f_k(X)$ of the characteristic polynomial

$$\det(\lambda I + X) = \sum_{k=0}^{r} f_k(X)\lambda^{r-k}.$$

The characteristic classes associated to $f_1(X),\dots,f_k(X)$ are the ***Chern classes*** of a principal $\mathrm{GL}(r,\mathbb{C})$-bundle. These Chern classes generalize the Chern classes of the frame bundle $\mathrm{Fr}(E)$ of a complex vector bundle E of rank r.

Example 32.4. If the Lie group G is $\mathrm{GL}(r,\mathbb{R})$, then by Theorem B.13 the ring of $\mathrm{Ad}(G)$-invariant polynomials on $\mathfrak{gl}(r,\mathbb{R})$ is also generated by the coefficients $f_k(X)$ of the characteristic polynomial

$$\det(\lambda I + X) = \sum_{k=0}^{r} f_k(X)\lambda^{r-k}.$$

The characteristic classes associated to $f_1(X),\dots,f_k(X)$ generalize the Pontrjagin classes of the frame bundle $\mathrm{Fr}(E)$ of a real vector bundle E of rank r. (For a real frame bundle the coefficients $f_k(\Omega)$ vanish for k odd.)

Problems

32.1. Polynomials on a vector space
Let V be a vector space with bases $e_1,\dots,e_n$ and $u_1,\dots,u_n$. Prove that if a function $f\colon V\to\mathbb{R}$ is a polynomial of degree k with respect to the basis $e_1,\dots,e_n$, then it is a polynomial of degree k with respect to the basis $u_1,\dots,u_n$. Thus, the notion of a polynomial of degree k on a vector space V is independent of the choice of a basis.

32.2. Chern–Weil forms
In this problem we keep the notations of this section. Let $e_1,\dots,e_n$ and $u_1,\dots u_n$ be two bases for the Lie algebra $\mathfrak{g}$ with dual bases $\alpha^1,\dots,\alpha^n$ and $\beta^1,\dots,\beta^n$, respectively. Suppose

$$\Omega = \sum \Omega^i e_i = \sum \Psi^j u_j$$

and

$$f = \sum a_I \alpha^{i_1}\cdots\alpha^{i_k} = \sum b_I \beta^{i_1}\cdots\beta^{i_k}.$$

Prove that

$$\sum a_I \Omega^{i_1}\wedge\cdots\wedge\Omega^{i_k} = \sum b_I \Psi^{i_1}\wedge\cdots\wedge\Psi^{i_k}.$$

This shows that the definition of $f(\Omega)$ is independent of the choice of basis for $\mathfrak{g}$.

32.3. Connection on $P\times I$
Show that the 1-form $\tilde{\omega}$ in (32.2) is a connection on $P\times I$.

32.4. Chern–Weil homomorphism
Show that the map $w\colon \mathrm{Inv}(\mathfrak{g})\to H^*(M)$ in (32.3) is an algebra homomorphism.

Appendices

§A Manifolds

This appendix is a review, mostly without proofs, of the basic notions in the theory of manifolds and differential forms. For more details, see [21].

A.1 Manifolds and Smooth Maps

We will be following the convention of classical differential geometry in which vector fields take on subscripts, differential forms take on superscripts, and coefficient functions can have either superscripts or subscripts depending on whether they are coefficient functions of vector fields or of differential forms. See [21, §4.7, p. 44] for a more detailed explanation of this convention.

A manifold is a higher-dimensional analogue of a smooth curve or surface. Its prototype is the Euclidean space $\mathbb{R}^n$, with coordinates $r^1, \dots, r^n$. Let U be an open subset of $\mathbb{R}^n$. A function $f = (f^1, \dots, f^m)\colon U \to \mathbb{R}^m$ is ***smooth*** on U if the partial derivatives $\partial^k f/\partial r^{j_1} \cdots \partial r^{j_k}$ exist on U for all integers $k \geq 1$ and all $j_1, \dots, j_k$. In this book we use the terms "smooth" and "C^∞" interchangeably.

A topological space M is ***locally Euclidean*** of dimension n if for every point p in M, there is a homeomorphism ϕ of a neighborhood U of p with an open subset of $\mathbb{R}^n$. Such a pair $(U, \phi\colon U \to \mathbb{R}^n)$ is called a ***coordinate chart*** or simply a ***chart***. If $p \in U$, then we say that (U, ϕ) is a ***chart about*** p. A collection of charts $\{(U_\alpha, \phi_\alpha\colon U_\alpha \to \mathbb{R}^n)\}$ is C^∞ ***compatible*** if for every α and β, the transition function

$$\phi_\alpha \circ \phi_\beta^{-1}\colon \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is C^∞. A collection of C^∞ compatible charts $\{(U_\alpha, \phi_\alpha\colon U_\alpha \to \mathbb{R}^n)\}$ that cover M is called a C^∞ ***atlas***. A C^∞ atlas is said to be ***maximal*** if it contains every chart that is C^∞ compatible with all the charts in the atlas.

L.W. Tu, *Differential Geometry*, Graduate Texts in Mathematics 275,
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Definition A.1. A ***topological manifold*** is a Hausdorff, second countable, locally Euclidean topological space. By "second countable," we mean that the space has a countable basis of open sets. A ***smooth*** or C^∞ ***manifold*** is a pair consisting of a topological manifold M and a maximal C^∞ atlas $\{(U_\alpha, \phi_\alpha)\}$ on M. In this book all manifolds will be smooth manifolds.

In the definition of a manifold, the Hausdorff condition excludes certain pathological examples, while the second countability condition guarantees the existence of a partition of unity, a useful technical tool that we will define shortly.

In practice, to show that a Hausdorff, second countable topological space is a smooth manifold it suffices to exhibit a C^∞ atlas, for by Zorn's lemma every C^∞ atlas is contained in a unique maximal atlas.

Example A.2. Let S^1 be the circle defined by $x^2 + y^2 = 1$ in $\mathbb{R}^2$, with open sets (see Figure A.1)

$$\begin{aligned} U_x^+ &= \{(x,y) \in S^1 \mid x > 0\}, \\ U_x^- &= \{(x,y) \in S^1 \mid x < 0\}, \\ U_y^+ &= \{(x,y) \in S^1 \mid y > 0\}, \\ U_y^- &= \{(x,y) \in S^1 \mid y < 0\}. \end{aligned}$$

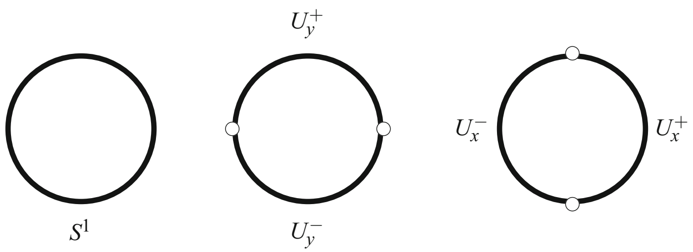


Fig. A.1. A C^∞ atlas on S^1.

Then $\{(U_x^+, y), (U_x^-, y), (U_y^+, x), (U_y^-, x)\}$ is a C^∞ atlas on S^1. For example, the transition function from

$$\text{the open interval }\ ]0,1[= x(U_x^+ \cap U_y^-) \to y(U_x^+ \cap U_y^-) =]-1,0[$$

is $y = -\sqrt{1-x^2}$, which is C^∞ on its domain.

A function $f\colon M \to \mathbb{R}^n$ on a manifold M is said to be ***smooth*** or C^∞ at $p \in M$ if there is a chart (U, ϕ) about p in the maximal atlas of M such that the function

$$f \circ \phi^{-1} \colon \mathbb{R}^m \supset \phi(U) \to \mathbb{R}^n$$

is C^∞. The function $f\colon M \to \mathbb{R}^n$ is said to be ***smooth*** or C^∞ on M if it is C^∞ at every point of M. Recall that an ***algebra*** over $\mathbb{R}$ is a vector space together with a bilinear map $\mu\colon A \times A \to A$, called ***multiplication***, such that under addition and multiplication, A becomes a ring. Under addition, multiplication, and scalar multiplication, the set of all C^∞ functions $f\colon M \to \mathbb{R}$ is an algebra over $\mathbb{R}$, denoted by $C^\infty(M)$.

A map $F\colon N \to M$ between two manifolds is ***smooth*** or C^∞ at $p \in N$ if there is a chart (U,ϕ) about p in N and a chart (V,ψ) about $F(p)$ in M with $V \supset F(U)$ such that the composite map $\psi \circ F \circ \phi^{-1}\colon \mathbb{R}^n \supset \phi(U) \to \psi(V) \subset \mathbb{R}^m$ is C^∞ at $\phi(p)$. It is ***smooth*** on N if it is smooth at every point of N. A smooth map $F\colon N \to M$ is called a ***diffeomorphism*** if it has a smooth inverse, i.e., a smooth map $G\colon M \to N$ such that $F \circ G = \mathbb{1}_M$ and $G \circ F = \mathbb{1}_N$.

A typical matrix in linear algebra is usually an $m \times n$ matrix, with m rows and n columns. Such a matrix represents a linear transformation $F\colon \mathbb{R}^n \to \mathbb{R}^m$. For this reason, we usually write a C^∞ map as $F\colon N \to M$, rather than $F\colon M \to N$.

A.2 Tangent Vectors

The derivatives of a function f at a point p in $\mathbb{R}^n$ depend only on the values of f in a small neighborhood of p. To make precise what is meant by a "small" neighborhood, we introduce the concept of the germ of a function. Decree two C^∞ functions $f\colon U \to \mathbb{R}$ and $g\colon V \to \mathbb{R}$ defined on neighborhoods U and V of p to be ***equivalent*** if there is a neighborhood W of p contained in both U and V such that f agrees with g on W. The equivalence class of $f\colon U \to \mathbb{R}$ is called the ***germ*** of f at p.

It is easy to verify that addition, multiplication, and scalar multiplication are well-defined operations on the set $C_p^\infty(M)$ of germs of C^∞ real-valued functions at p in M. These three operations make $C_p^\infty(M)$ into an algebra over $\mathbb{R}$.

Definition A.3. A ***point-derivation*** at a point p of a manifold M is a linear map $D\colon C_p^\infty(M) \to \mathbb{R}$ such that for any $f, g \in C_p^\infty(M)$,

$$D(fg) = (Df)g(p) + f(p)Dg.$$

A point-derivation at p is also called a ***tangent vector*** at p. The set of all tangent vectors at p is a vector space T_pM, called the ***tangent space*** of M at p.

Example A.4. If $r^1,\dots,r^n$ are the standard coordinates on $\mathbb{R}^n$ and $p \in \mathbb{R}^n$, then the usual partial derivatives

$$\left.\frac{\partial}{\partial r^1}\right|_p, \dots, \left.\frac{\partial}{\partial r^n}\right|_p$$

are tangent vectors at p that form a basis for the tangent space $T_p(\mathbb{R}^n)$.

At a point p in a coordinate chart $(U,\phi) = (U,x^1,\dots,x^n)$, where $x^i = r^i \circ \phi$ is the ith component of ϕ, we define the ***coordinate vectors*** $\partial/\partial x^i|_p \in T_pM$ by

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} f \circ \phi^{-1} \qquad \text{for each } f \in C_p^\infty(M).$$

If $F\colon N \to M$ is a C^∞ map, then at each point $p \in N$ its ***differential***

$$F_{*,p}\colon T_pN \to T_{F(p)}M, \tag{A.1}$$

is the linear map defined by

$$(F_{*,p}X_p)(h) = X_p(h \circ F)$$

for $X_p \in T_pN$ and $h \in C^\infty_{F(p)}(M)$. Usually the point p is clear from context and we may write F_* instead of $F_{*,p}$. It is easy to verify that if $F\colon N \to M$ and $G\colon M \to P$ are C^∞ maps, then for any $p \in N$,

$$(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p},$$

or, suppressing the points,

$$(G \circ F)_* = G_* \circ F_*.$$

A.3 Vector Fields

A ***vector field*** X on a manifold M is the assignment of a tangent vector $X_p \in T_pM$ to each point $p \in M$. At every p in a chart $(U, x^1, \dots, x^n)$, since the coordinate vectors $\partial/\partial x^i|_p$ form a basis of the tangent space T_pM, the vector X_p can be written as a linear combination

$$X_p = \sum_i a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p \quad \text{with } a^i(p) \in \mathbb{R}.$$

As p varies over U, the coefficients $a^i(p)$ become functions on U. The vector field X is said to be ***smooth*** or C^∞ if M has a C^∞ atlas such that on each chart $(U, x^1, \dots, x^n)$ of the atlas, the coefficient functions a^i in $X = \sum a^i \partial/\partial x^i$ are C^∞. We denote the set of all C^∞ vector fields on M by $\mathfrak{X}(M)$. It is a vector space under the addition of vector fields and scalar multiplication by real numbers. As a matter of notation, we write tangent vectors at p as $X_p, Y_p, Z_p \in T_pM$, or if the point p is understood from context, as $v_1, v_2, \dots, v_k \in T_pM$. As a shorthand, we sometimes write ∂_i for $\partial/\partial x^i$.

A ***frame of vector fields*** on an open set $U \subset M$ is a collection of vector fields $X_1, \dots, X_n$ on U such that at each point $p \in U$, the vectors $(X_1)_p, \dots, (X_n)_p$ form a basis for the tangent space T_pM. For example, in a coordinate chart $(U, x^1, \dots, x^n)$, the coordinate vector fields $\partial/\partial x^1, \dots, \partial/\partial x^n$ form a frame of vector fields on U.

A C^∞ vector field X on a manifold M gives rise to a linear operator on the vector space $C^\infty(M)$ of C^∞ functions on M by the rule

$$(Xf)(p) = X_pf \quad \text{for } f \in C^\infty(M) \text{ and } p \in M.$$

To show that Xf is a C^∞ function on M, it suffices to write X in terms of local coordinates $x^1, \dots, x^n$ in a neighborhood of p, say $X = \sum a^i \partial/\partial x^i$. Since X is assumed C^∞, all the coefficients a^i are C^∞. Therefore, if f is C^∞, then $Xf = \sum a^i \partial f/\partial x^i$ is also.

The ***Lie bracket*** of two vector fields $X,Y \in \mathfrak{X}(M)$ is the vector field $[X,Y]$ defined by

$$[X,Y]_p f = X_p(Yf) - Y_p(Xf) \quad \text{for } p \in M \text{ and } f \in C_p^\infty(M). \tag{A.2}$$

So defined, $[X,Y]_p$ is a point-derivation at p (Problem A.3(a)) and therefore $[X,Y]$ is indeed a vector field on M. The formula for $[X,Y]$ in local coordinates (Problem A.3(b)) shows that if X and Y are C^∞, then so is $[X,Y]$.

If $f\colon N \to M$ is a C^∞ map, its differential $f_{*,p}\colon T_pN \to T_{f(p)}M$ pushes forward a tangent vector at a point in N to a tangent vector in M. It should be noted, however, that in general there is no push-forward map $f_*\colon \mathfrak{X}(N) \to \mathfrak{X}(M)$ for vector fields. For example, when f is not one-to-one, say $f(p) = f(q)$ for $p \neq q$ in N, it may happen that for some $X \in \mathfrak{X}(N)$, $f_{*,p}X_p \neq f_{*,q}X_q$; in this case, there is no way to define f_*X so that $(f_*X)_{f(p)} = f_{*,p}X_p$ for all $p \in N$. Similarly, if $f\colon N \to M$ is not onto, then there is no natural way to define f_*X at a point of M not in the image of f. Of course, if $f\colon N \to M$ is a diffeomorphism, then the pushforward $f_*\colon \mathfrak{X}(N) \to \mathfrak{X}(M)$ is well defined.

A.4 Differential Forms

For $k \geq 1$, a ***k-form*** or a ***form of degree*** k on M is the assignment to each p in M of an alternating k-linear function

$$\omega_p\colon \underbrace{T_pM \times \cdots \times T_pM}_{k \text{ copies}} \to \mathbb{R}.$$

Here "alternating" means that for every permutation σ of the set $\{1,2,\ldots,k\}$ and $v_1,\ldots,v_k \in T_pM$,

$$\omega_p(v_{\sigma(1)},\ldots,v_{\sigma(k)}) = (\operatorname{sgn}\sigma)\,\omega_p(v_1,\ldots,v_k), \tag{A.3}$$

where $\operatorname{sgn}\sigma$, the sign of the permutation σ, is ± 1 depending on whether σ is even or odd. We define a 0-form to be the assignment of a real number to each $p \in M$; in other words, a 0-form on M is simply a real-valued function on M. When $k = 1$, the condition of being alternating is vacuous. Thus, a 1-form on M is the assignment of a linear function $\omega_p\colon T_pM \to \mathbb{R}$ to each p in M. For $k < 0$, a k-form is 0 by definition.

An alternating k-linear function on a vector space V is also called a ***k-covector*** on V. As above, a 0-covector is a constant and a 1-covector on V is a linear function $f\colon V \to \mathbb{R}$. Let $A_k(V)$ be the vector space of all k-covectors on V. Then $A_0(V) = \mathbb{R}$ and $A_1(V) = V^\vee := \operatorname{Hom}(V,\mathbb{R})$, the dual vector space of V. In this language a k-form on M is the assignment of a k-covector $\omega_p \in A_k(T_pM)$ to each point p in M.

Let S_k be the group of all permutations of $\{1,2,\ldots,k\}$. A (k,ℓ)***-shuffle*** is a permutation $\sigma \in S_{k+\ell}$ such that

$$\sigma(1) < \cdots < \sigma(k) \text{ and } \sigma(k+1) < \cdots < \sigma(k+\ell).$$

The ***wedge product*** of a k-covector α and an ℓ-covector β on a vector space V is by definition the $(k+\ell)$-linear function

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+\ell}) = \sum (\operatorname{sgn} \sigma) \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}), \quad \text{(A.4)}$$

where the sum runs over all (k,ℓ)-shuffles. For example, if α and β are 1-covectors, then

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1).$$

The wedge of a 0-covector, i.e., a constant c, with another covector ω is simply scalar multiplication. In this case, in keeping with the traditional notation for scalar multiplication, we often replace the wedge by a dot or even by nothing: $c \wedge \omega = c \cdot \omega = c\omega$.

The wedge product $\alpha \wedge \beta$ is a $(k+\ell)$-covector; moreover, the wedge operation $\wedge$ is bilinear, associative, and anticommutative in its two arguments. ***Anticommutativity*** means that

$$\alpha \wedge \beta = (-1)^{\deg \alpha \deg \beta} \beta \wedge \alpha.$$

Proposition A.5. *If $\alpha^1, \dots, \alpha^n$ is a basis for the 1-covectors on a vector space V, then a basis for the k-covectors on V is the set*

$$\{\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\}.$$

A k-tuple of integers $I = (i_1, \dots, i_k)$ is called a ***multi-index***. If $i_1 \le \cdots \le i_k$, we call I an ***ascending multi-index***, and if $i_1 < \cdots < i_k$, we call I a ***strictly ascending multi-index***. To simplify the notation, we will write $\alpha^I = \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}$.

As noted earlier, for a point p in a coordinate chart $(U, x^1, \dots, x^n)$, a basis for the tangent space T_pM is

$$\left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p.$$

Let $(dx^1)_p, \dots, (dx^n)_p$ be the dual basis for the cotangent space $A_1(T_pM) = T_p^*M$, i.e.,

$$(dx^i)_p \left(\left.\frac{\partial}{\partial x^j}\right|_p \right) = \delta^i_j.$$

By Proposition A.5, if ω is a k-form on M, then at each $p \in U$, ω_p is a linear combination:

$$\omega_p = \sum_I a_I(p)(dx^I)_p = \sum_I a_I(p)(dx^{i_1})_p \wedge \cdots \wedge (dx^{i_k})_p.$$

We say that the k-form ω is ***smooth*** if M has an atlas $\{(U, x^1, \dots, x^n)\}$ such that on each U, the coefficients $a_I \colon U \to \mathbb{R}$ of ω are smooth. By ***differential k-forms***, we will mean smooth k-forms on a manifold.

A ***frame of differential k-forms*** on an open set $U \subset M$ is a collection of differential k-forms $\omega_1, \dots, \omega_r$ on U such that at each point $p \in U$, the k-covectors $(\omega_1)_p, \dots, (\omega_r)_p$ form a basis for the vector space $A_k(T_pM)$ of k-covectors on the

tangent space at p. For example, on a coordinate chart $(U, x^1, \dots, x^n)$, the k-forms $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, $1 \le i_1 < \cdots < i_k \le n$, constitute a frame of differential k-forms on U.

A subset B of a left R-module V is called a ***basis*** if every element of V can be written uniquely as a finite linear combination $\sum r_i b_i$, where $r_i \in R$ and $b_i \in B$. An R-module with a basis is said to be ***free***, and if the basis is finite with n elements, then the free R-module is said to be of ***rank*** n. It can be shown that if a free R-module has a finite basis, then any two bases have the same number of elements, so that the rank is well defined. We denote the rank of a free R-module V by $\operatorname{rk} V$.

Let $\Omega^k(M)$ denote the vector space of C^∞ k-forms on M and let

$$\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M).$$

If $(U, x^1, \dots, x^n)$ is a coordinate chart on M, then $\Omega^k(U)$ is a free module over $C^\infty(U)$ of rank $\binom{n}{k}$, with basis dx^I as above.

An algebra A is said to be ***graded*** if it can be written as a direct sum $A = \bigoplus_{k=0}^\infty A_k$ of vector spaces such that under multiplication, $A_k \cdot A_\ell \subset A_{k+\ell}$. The wedge product $\wedge$ makes $\Omega^*(M)$ into an anticommutative graded algebra over $\mathbb{R}$.

A.5 Exterior Differentiation on a Manifold

An ***exterior derivative*** on a manifold M is a linear operator $d\colon \Omega^*(M) \to \Omega^*(M)$, satisfying the following three properties:

(1) d is an antiderivation of degree 1, i.e., d increases the degree by 1 and for $\omega \in \Omega^k(M)$ and $\tau \in \Omega^\ell(M)$,

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^k \omega \wedge d\tau;$$

(2) $d^2 = d \circ d = 0$;
(3) on a 0-form $f \in C^\infty(M)$,

$$(df)_p(X_p) = X_p f \text{ for } p \in M \text{ and } X_p \in T_pM.$$

By induction the antiderivation property (1) extends to more than two factors; for example,

$$d(\omega \wedge \tau \wedge \eta) = d\omega \wedge \tau \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\tau \wedge \eta + (-1)^{\deg \omega \wedge \tau} \omega \wedge \tau \wedge d\eta.$$

The existence and uniqueness of an exterior derivative on a general manifold is established in [21, Section 19]. To develop some facility with this operator, we will examine the case when M is covered by a single coordinate chart $(U, x^1, \dots, x^n)$. This case can be used to define and compute locally on a manifold.

Proposition A.6. *Let $(U, x^1, \dots, x^n)$ be a coordinate chart. Suppose $d\colon \Omega^*(U) \to \Omega^*(U)$ is an exterior derivative. Then*

(i) *for any* $f \in \Omega^0(U)$,

$$df = \sum \frac{\partial f}{\partial x^i}\, dx^i;$$

(ii) $d(dx^I) = 0$;
(iii) *for any* $a_I\, dx^I \in \Omega^k(M)$, $d(a_I\, dx^I) = da_I \wedge dx^I$.

Proof. (i) Since $(dx^1)_p, \dots, (dx^n)_p$ is a basis of 1-covectors at each point $p \in U$, there are constants $a_i(p)$ such that

$$(df)_p = \sum a_i(p)\,(dx^i)_p.$$

Suppressing p, we may write

$$df = \sum a_i\, dx^i.$$

Applying both sides to the vector field $\partial/\partial x^i$ gives

$$df\left(\frac{\partial}{\partial x^j}\right) = \sum_i a_i\, dx^i \left(\frac{\partial}{\partial x^j}\right) = \sum_i a_i \delta^i_j = a_j.$$

On the other hand, by property (3) of d,

$$df\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial x^j}(f).$$

Hence, $a_j = \partial f/\partial x^j$ and $df = \sum(\partial f/\partial x^j)\, dx^j$.

(ii) By the antiderivation property of d,

$$\begin{aligned} d(dx^I) = d(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) &= \sum_j (-1)^{j-1} dx^{i_1} \wedge \cdots \wedge ddx^{i_j} \wedge \cdots \wedge dx^{i_k} \\ &= 0 \qquad \text{since } d^2 = 0. \end{aligned}$$

(iii) By the antiderivation property of d,

$$\begin{aligned} d\left(a_I\, dx^I\right) &= da_I \wedge dx^I + a_I\, d(dx^I) \\ &= da_I \wedge dx^I \qquad \text{since } d(dx^I) = 0. \end{aligned}$$ □

Proposition A.6 proves the uniqueness of exterior differentiation on a coordinate chart $(U, x^1, \dots, x^n)$. To prove its existence, we define d by two of the formulas of Proposition A.6:

(i) if $f \in \Omega^0(U)$, then $df = \sum(\partial f/\partial x^i)\, dx^i$;
(iii) if $\omega = \sum a_I\, dx^I \in \Omega^k(U)$ for $k \geq 1$, then $d\omega = \sum da_I \wedge dx^I$.

Next we check that so defined, d satisfies the three properties of exterior differentiation.

(1) For $\omega \in \Omega^k(U)$ and $\tau \in \Omega^\ell(U)$,

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^k \omega \wedge d\tau. \tag{A.5}$$

Proof. Suppose $\omega = \sum a_I\, dx^I$ and $\tau = \sum b_J\, dx^J$. On functions, $d(fg) = (df)g + f(dg)$ is simply another manifestation of the ordinary product rule, since

$$\begin{aligned} d(fg) &= \sum \frac{\partial}{\partial x^i}(fg)\, dx^i \\ &= \sum \left(\frac{\partial f}{\partial x^i} g + f \frac{\partial g}{\partial x^i} \right) dx^i \\ &= \left(\sum \frac{\partial f}{\partial x^i}\, dx^i \right) g + f \sum \frac{\partial g}{\partial x^i}\, dx^i \\ &= (df)\, g + f\, dg. \end{aligned}$$

Next suppose $k \geq 1$. Since d is linear and $\wedge$ is bilinear over $\mathbb{R}$, we may assume that $\omega = a_I\, dx^I$ and $\tau = b_J\, dx^J$, each consisting of a single term. Then

$$\begin{aligned} d(\omega \wedge \tau) &= d(a_I b_J\, dx^I \wedge dx^J) \\ &= d(a_I b_J) \wedge dx^I \wedge dx^J \quad \text{(definition of } d\text{)} \\ &= (da_I) b_J \wedge dx^I \wedge dx^J + a_I\, db_J \wedge dx^I \wedge dx^J \\ &\qquad \text{(by the degree 0 case)} \\ &= da_I \wedge dx^I \wedge b_J\, dx^J + (-1)^k a_I\, dx^I \wedge db_J \wedge dx^J \\ &= d\omega \wedge \tau + (-1)^k \omega \wedge d\tau. \end{aligned}$$

□

(2) $d^2 = 0$ on $\Omega^k(U)$.

Proof. This is a consequence of the fact that the mixed partials of a function are equal. For $f \in \Omega^0(U)$,

$$d^2 f = d\left(\sum_{i=1}^n \frac{\partial f}{\partial x^i}\, dx^i \right) = \sum_{j=1}^n \sum_{i=1}^n \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i.$$

In this double sum, the factors $\partial^2 f / \partial x^j \partial x^i$ are symmetric in i, j, while $dx^j \wedge dx^i$ are skew-symmetric in i, j. Hence, for each pair $i < j$ there are two terms

$$\frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j, \qquad \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i$$

that add up to zero. It follows that $d^2 f = 0$.

For $\omega = \sum a_I\, dx^I \in \Omega^k(U)$, where $k \geq 1$,

$$\begin{aligned} d^2 \omega &= d\left(\sum da_I \wedge dx^I \right) \qquad \text{(by the definition of } d\omega\text{)} \\ &= \sum (d^2 a_I) \wedge dx^I + da_I \wedge d(dx^I) \\ &= 0. \end{aligned}$$

In this computation, $d^2 a_I = 0$ by the degree 0 case, and $d(dx^I) = 0$ follows as in the proof of Proposition A.6(ii) by the antiderivation property and the degree 0 case. □

(3) For f a C^∞ function and X a C^∞ vector field on $(U, x^1, \dots, x^n)$, $(df)(X) = Xf$.

Proof. Suppose $X = \sum a^j \partial/\partial x^j$. Then

$$(df)(X) = \left(\sum \frac{\partial f}{\partial x^i} dx^i\right)\left(\sum a^j \frac{\partial}{\partial x^j}\right) = \sum a^i \frac{\partial f}{\partial x^i} = Xf. \qquad \square$$

A.6 Exterior Differentiation on $\mathbb{R}^3$

On $\mathbb{R}^3$ with coordinates x, y, z, every smooth vector field X is uniquely a linear combination

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$$

with coefficient functions $a, b, c \in C^\infty(\mathbb{R}^3)$. Thus, the vector space $\mathfrak{X}(\mathbb{R}^3)$ of smooth vector fields on $\mathbb{R}^3$ is a free module of rank 3 over $C^\infty(\mathbb{R}^3)$ with basis $\{\partial/\partial x, \partial/\partial y, \partial/\partial z\}$. Similarly, $\Omega^3(\mathbb{R}^3)$ is a free module of rank 1 over $C^\infty(\mathbb{R}^3)$ with basis $\{dx \wedge dy \wedge dz\}$, while $\Omega^1(\mathbb{R}^3)$ and $\Omega^2(\mathbb{R}^3)$ are free modules of rank 3 over $C^\infty(\mathbb{R}^3)$ with bases $\{dx, dy, dz\}$ and $\{dy \wedge dz,\ dz \wedge dx,\ dx \wedge dy\}$, respectively. So the following identifications are possible:

$$\begin{array}{ccccc} \text{functions} & = & \text{0-forms} & \longleftrightarrow & \text{3-forms} \\ f & = & f & \longleftrightarrow & f\, dx \wedge dy \wedge dz \end{array}$$

and

$$\begin{array}{ccccc} \text{vector fields} & \leftrightarrow & \text{1-forms} & \leftrightarrow & \text{2-forms} \\ X = \langle a, b, c \rangle & \leftrightarrow & a\,dx + b\,dy + c\,dz & \leftrightarrow & a\,dy \wedge dz + b\,dz \wedge dx + c\,dx \wedge dy. \end{array}$$

We will write $f_x = \partial f/\partial x$, $f_y = \partial f/\partial y$, and $f_z = \partial f/\partial z$. On functions,

$$df = f_x\,dx + f_y\,dy + f_z\,dz.$$

On 1-forms,

$$d(a\,dx + b\,dy + c\,dz) = (c_y - b_z)\,dy \wedge dz - (c_x - a_z)\,dz \wedge dx + (b_x - a_y)\,dx \wedge dy.$$

On 2-forms,

$$d(a\,dy \wedge dz + b\,dz \wedge dx + c\,dx \wedge dy) = (a_x + b_y + c_z)\,dx \wedge dy \wedge dz.$$

Identifying forms with vector fields and functions, we have the following correspondences:

$$\begin{aligned} d(\text{0-form}) &\longleftrightarrow \text{gradient of a function,} \\ d(\text{1-form}) &\longleftrightarrow \text{curl of a vector field,} \\ d(\text{2-form}) &\longleftrightarrow \text{divergence of a vector field.} \end{aligned}$$

A.7 Pullback of Differential Forms

Unlike vector fields, which in general cannot be pushed forward under a C^∞ map, differential forms can always be pulled back. Let $F\colon N \to M$ be a C^∞ map. The ***pullback*** of a C^∞ function f on M is the C^∞ function $F^*f := f \circ F$ on N. This defines the pullback on C^∞ 0-forms. For $k > 0$, the ***pullback*** of a k-form ω on M is the k-form $F^*\omega$ on N defined by

$$(F^*\omega)_p(v_1,\dots,v_k) = \omega_{F(p)}(F_{*,p}v_1,\dots,F_{*,p}v_k)$$

for $p \in N$ and $v_1,\dots,v_k \in T_pM$. From this definition, it is not obvious that the pullback $F^*\omega$ of a C^∞ form ω is C^∞. To show this, we first derive a few basic properties of the pullback.

Proposition A.7. *Let $F\colon N \to M$ be a C^∞ map of manifolds. If ω and τ are k-forms and σ is an ℓ-form on M, then*

(i) $F^*(\omega+\tau) = F^*\omega + F^*\tau$;
(ii) *for any real number a, $F^*(a\omega) = aF^*\omega$;*
(iii) $F^*(\omega \wedge \sigma) = F^*\omega \wedge F^*\sigma$;
(iv) *for any C^∞ function h on M, $dF^*h = F^*dh$.*

Proof. The first three properties (i), (ii), (iii) follow directly from the definitions. To prove (iv), let $p \in N$ and $X_p \in T_pN$. Then

$$\begin{aligned}(dF^*h)_p(X_p) &= X_p(F^*h) && \text{(property (3) of } d)\\ &= X_p(h\circ F) && \text{(definition of } F^*h)\end{aligned}$$

and

$$\begin{aligned}(F^*dh)_p(X_p) &= (dh)_{F(p)}(F_{*,p}X_p) && \text{(definition of } F^*)\\ &= (F_{*,p}X_p)h && \text{(property (3) of } d)\\ &= X_p(h\circ F). && \text{(definition of } F_{*,p})\end{aligned}$$

Hence,

$$dF^*h = F^*dh. \qquad \square$$

We now prove that the pullback of a C^∞ form is C^∞. On a coordinate chart $(U,x^1,\dots,x^n)$ in M, a C^∞ k-form ω can be written as a linear combination

$$\omega = \sum a_I\, dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where the coefficients a_I are C^∞ functions on U. By the preceding proposition,

$$\begin{aligned}F^*\omega &= \sum (F^*a_I)\, d(F^*x^{i_1}) \wedge \cdots \wedge d(F^*x^{i_k})\\ &= \sum (a_I\circ F)\, d(x^{i_1}\circ F) \wedge \cdots \wedge d(x^{i_k}\circ F),\end{aligned}$$

which shows that $F^*\omega$ is C^∞, because it is a sum of products of C^∞ functions and C^∞ 1-forms.

Proposition A.8. *Suppose $F\colon N \to M$ is a smooth map. On C^∞ k-forms, $dF^* = F^*d$.*

Proof. Let $\omega \in \Omega^k(M)$ and $p \in M$. Choose a chart $(U, x^1, \dots, x^n)$ about p in M. On U,

$$\omega = \sum a_I \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

As computed above,

$$F^*\omega = \sum (a_I \circ F)\, d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F).$$

Hence,

$$\begin{aligned}
dF^*\omega &= \sum d(a_I \circ F) \wedge d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F) \\
&= \sum d(F^* a_I) \wedge d(F^* x^{i_1}) \wedge \cdots \wedge d(F^* x^{i_k}) \\
&= \sum F^* da_I \wedge F^* dx^{i_1} \wedge \cdots \wedge F^* dx^{i_k} \\
&\quad (dF^* = F^*d \text{ on functions by Prop. A.7(iv)}) \\
&= \sum F^*(da_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\
&\quad (F^* \text{ preserves the wedge product by Prop. A.7(iii)}) \\
&= F^* d\omega.
\end{aligned}$$

□

In summary, for any C^∞ map $F\colon N \to M$, the pullback map $F^*\colon \Omega^*(M) \to \Omega^*(N)$ is an algebra homomorphism that commutes with the exterior derivative d.

Example A.9 (*Pullback under the inclusion map of an immersed submanifold*). Let N and M be manifolds. A C^∞ map $f\colon N \to M$ is called an ***immersion*** if for all $p \in N$, the differential $f_{*,p}\colon T_pN \to T_{f(p)}M$ is injective. A subset S of M with a manifold structure such that the inclusion map $i\colon S \hookrightarrow M$ is an immersion is called an ***immersed submanifold*** of M. An example is the image of a line with irrational slope in the torus $\mathbb{R}^2/\mathbb{Z}^2$. An immersed submanifold need not have the subspace topology.

If $\omega \in \Omega^k(M)$, $p \in S$, and $v_1, \dots, v_k \in T_pS$, then by the definition of the pullback,

$$(i^*\omega)_p(v_1, \dots, v_k) = \omega_{i(p)}(i_*v_1, \dots, i_*v_k) = \omega_p(v_1, \dots, v_k).$$

Thus, the pullback of ω under the inclusion map $i\colon S \hookrightarrow M$ is simply the restriction of ω to the submanifold S. We also adopt the more suggestive notation $\omega|_S$ for $i^*\omega$.

Problems

A.1. Connected components

(a) The ***connected component*** of a point p in a topological space S is the largest connected subset of S containing p. Show that the connected components of a manifold are open.

(b) Let $\mathbb{Q}$ be the set of rational numbers considered as a subspace of the real line $\mathbb{R}$. Show that the connected component of $p \in \mathbb{Q}$ is the singleton set $\{p\}$, which is not open in $\mathbb{Q}$. Which condition in the definition of a manifold does $\mathbb{Q}$ violate?

A.2. Path-connectedness versus connectedness
A topological space S is said to be ***locally path-connected*** at a point $p \in S$ if for every neighborhood U of p, there is a path-connected neighborhood V of p such that $V \subset U$. The space S is ***locally path-connected*** if it is locally path-connected at every point $p \in S$. A ***path component*** of S is a maximal path-connected subset of S.

(a) Prove that in a locally path-connected space S, every path component is open.
(b) Prove that a locally path-connected space is path-connected if and only if it is connected.

A.3. The Lie bracket
Let X and Y be C^∞ vector fields on a manifold M, and p a point in M.

(a) Define $[X,Y]_p$ by (A.2). Show that for $f,g \in C_p^\infty(M)$,

$$[X,Y]_p(fg) = ([X,Y]_p f)g(p) + f(p)([X,Y]_p g).$$

Thus, $[X,Y]_p$ is a tangent vector at p.
(b) Suppose $X = \sum a^i \partial_i$ and $Y = \sum b^j \partial_j$ in a coordinate neighborhood $(U, x^1, \ldots, x^n)$ of p in M. Prove that

$$[X,Y] = \sum_{i,j} (a^j \partial_j b^i - b^j \partial_j a^i) \partial_i.$$

§B Invariant Polynomials

Let $X = [x^i_j]$ be an $r \times r$ matrix with indeterminate entries x^i_j. Over any field F, a polynomial $P(X) \in F[x^i_j]$ in the r^2 variables x^i_j is said to be ***invariant*** if for all invertible matrices A in the general linear group $\mathrm{GL}(r,F)$, we have

$$P(A^{-1}XA) = P(X).$$

Over the field of real or complex numbers, there are two sets of obviously invariant polynomials: (1) the coefficients $f_i(X)$ of the characteristic polynomial of $-X$:

$$\det(\lambda I + X) = \sum_{k=0}^{r} f_k(X)\lambda^{r-k},$$

and (2) the trace polynomials $\Sigma_k(X) = \mathrm{tr}(X^k)$. This appendix contains results on invariant polynomials needed in the sections on characteristic classes. We discuss first the distinction between polynomials and polynomial functions. Then we show that a polynomial identity with integer coefficients that holds over the reals holds over any commutative ring with 1. This is followed by the theorem that over the field of real or complex numbers, the ring of invariant polynomials is generated by the coefficients of the characteristic polynomial of $-X$. Finally, we prove Newton's identity relating the elementary symmetric polynomials to the power sums. As a corollary, the ring of invariant polynomials over $\mathbb{R}$ or $\mathbb{C}$ can also be generated by the trace polynomials.

B.1 Polynomials Versus Polynomial Functions

Let R be a commutative ring with identity 1. A ***polynomial*** in n variables over R is an element of the R-algebra $R[x_1,\dots,x_n]$, where $x_1, \dots, x_n$ are indeterminates. A polynomial $P(x) \in R[x_1,\dots,x_n]$ defines a function $\hat{P}\colon R^n \to R$ by evaluation. Hence, if $\mathrm{Fun}(R^n,R)$ denotes the R-algebra of functions from R^n to R, then there is a map

$$\begin{aligned} \varepsilon\colon R[x_1,\dots,x_n] &\to \mathrm{Fun}(R^n,R), \\ P &\mapsto \hat{P}. \end{aligned}$$

By the definition of addition and multiplication of functions, the map ε is clearly an R-algebra homomorphism. An element in the image of ε is called a ***polynomial function*** over R.

Example. Let F be the field $\mathbb{Z}/p\mathbb{Z}$ for a prime p. The set $F^\times = (\mathbb{Z}/p\mathbb{Z})^\times$ of nonzero elements of F forms a group of order $p-1$ under multiplication. By Lagrange's theorem from group theory, $x^{p-1} = 1$ for all $x \in F^\times$. Therefore, allowing $x = 0$, we get $x^p = x$ for all $x \in F$. Thus, although x^p and x are distinct polynomials in $F[x]$, they give rise to the same polynomial functions in $\mathrm{Fun}(F^n,F)$. In this example, the map

$$\varepsilon\colon F[x] \to \mathrm{Fun}(F,F)$$

is not injective.

Proposition B.1. *If F is an infinite field (of any characteristic), then a polynomial function $\hat{P}\colon F^n \to F$ is the zero function if and only if P is the zero polynomial; i.e., the map*

$$\varepsilon\colon F[x_1,\dots,x_n] \to \mathrm{Fun}(F^n,F)$$

is injective.

Proof. If P is the zero polynomial, then of course $\hat{P}$ is the zero function. We prove the converse by induction on n. Consider first the case $n = 1$. Suppose $P(x)$ is a polynomial of degree m in x such that $\hat{P}$ is the zero function. Since a nonzero polynomial of degree m can have at most m zeros, and $P(x)$ vanishes on an infinite field, $P(x)$ must be the zero polynomial.

Next, we make the induction hypothesis that ε is injective whenever the number of variables is $\leq n-1$. Let

$$P(x_1,\dots,x_{n-1},x_n) = \sum_{k=0}^{m} P_k(x_1,\dots,x_{n-1})x_n^k,$$

where $P_k(x_1,\dots,x_{n-1})$ are polynomials in $n-1$ variables. Suppose $\hat{P}$ is the zero function on F^n. Fix $(a_1,\dots,a_{n-1}) \in F^{n-1}$. Then $P(a_1,\dots,a_{n-1},x_n)$ is a polynomial in x_n, and $\hat{P}(a_1,\dots,a_{n-1},x_n) \in \mathrm{Fun}(F,F)$ is the zero function. By the one-variable case, $P(a_1,\dots,a_{n-1},x_n)$ is the zero polynomial. It follows that all its coefficients $P_k(a_1,\dots,a_{k-1})$ are zero. Since $(a_1,\dots,a_{n-1})$ is an arbitrary point of F^{n-1}, the polynomial function $\hat{P}_k$ is the zero function on F^{n-1}. By the induction hypothesis, the P_k's are all zero polynomials. Hence, $P(x_1,\dots,x_{n-1},x_n) = \sum_{k=0}^{m} P_k(x_1,\dots,x_{n-1})x_n^k$ is the zero polynomial. □

Thus, for an infinite field F, polynomial functions on F^n may be identified with polynomials in $F[x_1,\dots,x_n]$.

B.2 Polynomial Identities

Using Proposition B.1, we can derive the following principle by which a polynomial identity over one field can imply the same identity over any commutative ring with 1. Contrary to expectation, in this principle a special case implies a general case.

Proposition B.2 (Principle of extension of algebraic identities). *If a polynomial $P(x) \in \mathbb{Z}[x_1,\dots,x_n]$ with integer coefficients is identically zero when evaluated on a field F of characteristic zero, then it is identically zero when evaluated on any commutative ring R with* 1.

Proof. Because F has characteristic zero, the canonical map

$$\mathbb{Z} \to F, \quad k \mapsto k\cdot 1,$$

is an injection. The injection $\mathbb{Z} \to F$ induces an injection $\mathbb{Z}[x_1,\dots,x_n] \to F[x_1,\dots,x_n]$ of polynomial rings, so that $P(x) \in \mathbb{Z}[x_1,\dots,x_n]$ may be viewed as a polynomial

over F. By hypothesis, the polynomial function $\hat{P}$ on F^n is identically zero. Since F is an infinite field, we conclude from Proposition B.1 that $P(x)$ is the zero polynomial in $\mathbb{Z}[x_1,\dots,x_n]$.

If R is a commutative ring with 1, then there is a canonical map

$$\mathbb{Z} \to R, \quad k \mapsto k \cdot 1,$$

that induces a canonical map $\mathbb{Z}[x_1,\dots,x_n] \to R[x_1,\dots,x_n]$, which need not be an injection, but under which the zero polynomial $P(x)$ in $\mathbb{Z}[x]$ goes to the zero polynomial in $R[x]$. This means that $P(x)$ will be identically zero when evaluated on R. □

Example B.3 (*A determinantal identity*). Let a^i_j, b^i_j, $1 \le i, j \le r$, be indeterminates, and let $P(A,B)$ be the polynomial

$$P(A,B) = \det(AB) - \det(A)\det(B) \in \mathbb{Z}[a^i_j, b^i_j].$$

It is well known that if A and B are real $r \times r$ matrices, then $P(A,B)$ is zero. By the principle of extension of algebraic identities (Proposition B.2), for any commutative ring R with 1 and for all $A,B \in R^{r\times r}$,

$$P(A,B) = \det(AB) - \det(A)\det(B) = 0.$$

Thus, the truth of the identity $\det(AB) = \det(A)\det(B)$ over $\mathbb{R}$ implies its truth over any commutative ring R with 1.

In particular, over any commutative ring R with 1, if a matrix $A \in R^{r\times r}$ is invertible in $R^{r\times r}$, then $\det(A^{-1}) = (\det A)^{-1}$.

B.3 Invariant Polynomials on $\mathfrak{gl}(r,F)$

Let F be a field, and $\mathfrak{gl}(r,F) = F^{r\times r}$ the F-vector space of $r \times r$ matrices with entries in F. A ***polynomial*** $P(X)$ on $\mathfrak{gl}(r,F)$ is an element of the commutative F-algebra $F[x^i_j]$, i.e., it is a polynomial in the indeterminates x^i_j, $1 \le i, j \le r$. An invertible matrix $A \in \mathrm{GL}(r,F)$ acts on polynomials on $\mathfrak{gl}(r,F)$ by

$$(A \cdot P)(X) = P(A^{-1}XA).$$

A polynomial $P(X)$ on $\mathfrak{gl}(r,F)$ is said to be ***invariant*** if

$$P(A^{-1}XA) = P(X) \quad \text{for all } A \in \mathrm{GL}(r,F).$$

Example B.4 (*Determinant*). Let F be any field and $R = F[x^i_j]$ the polynomial algebra over F in $r \times r$ indeterminates. If $A \in GL(r,F)$, then

$$A \in F^{r\times r} \subset R^{r\times r}$$

and $X = [x^i_j]$ is a matrix of indeterminates in $R^{r\times r}$, so by Example B.3,

$$\det(A^{-1}XA) = (\det A)^{-1} \det X \det A = \det X.$$

Therefore, $\det(X) \in F[x^i_j]$ is an invariant polynomial on $\mathfrak{gl}(r,F)$.

Although an invariant polynomial on $\mathfrak{gl}(r,F)$ applies a priori to matrices with entries in F, it can in fact be applied to any commutative F-algebra R with identity.

Proposition B.5. *Suppose $P(X) \in F[x^i_j]$ is an invariant polynomial on $\mathfrak{gl}(r,F)$, and R is a commutative F-algebra with identity* 1. *Then for any $A \in \mathrm{GL}(r,F)$ and $X \in R^{r\times r}$,*

$$P(A^{-1}XA) = P(X).$$

Proof. Fix an invertible matrix $A \in \mathrm{GL}(r,F)$. Since $P(X)$ is invariant on $\mathfrak{gl}(r,F)$, the polynomial

$$P_A(X) := P(A^{-1}XA) - P(X) \in F[x^i_j]$$

is by definition the zero polynomial. If R is a commutative F-algebra with identity 1, then the canonical map $F \to R$, $f \mapsto f \cdot 1$, is injective and induces an injection $F[x^i_j] \hookrightarrow R[x^i_j]$, under which the zero polynomial $P_A(X)$ maps to the zero polynomial. Therefore, $P_A(X) = P(A^{-1}XA) - P(X)$ is the zero polynomial in $R[x^i_j]$. It follows that for any $X \in R^{r\times r}$,

$$P(A^{-1}XA) = P(X). \qquad \square$$

Let λ, x^i_j be indeterminates. Then

$$\begin{aligned}\det(\lambda I + X) &= \lambda^r + f_1(X)\lambda^{r-1} + \cdots f_{r-1}(X)\lambda + f_r(X)\\ &= \sum_{k=0}^{r} f_k(X)\lambda^{r-k}.\end{aligned}$$

The polynomials $f_k(X) \in \mathbb{Z}[x^i_j]$ have integer coefficients. For any field F, the canonical map $\mathbb{Z} \to F$ is not necessarily injective, for example, if the field F has positive characteristic. So the induced map $\varphi\colon \mathbb{Z}[x^i_j] \to F[x^i_j]$ is not necessarily injective. Although it is possible to define the polynomials f_k over any field, we will restrict ourselves to fields F of characteristic zero, for in this case the canonical maps $\mathbb{Z} \to F$ and $\mathbb{Z}[x^i_j] \to F[x^i_j]$ are injective, so we may view the $f_k(X)$'s in $\mathbb{Z}[x^i_j]$ as polynomials over F.

Proposition B.6. *Let F be a field of characteristic zero. Then the coefficients $f_k(X)$ of the characteristic polynomial* $\det(\lambda I + X)$ *are invariant polynomials on* $\mathfrak{gl}(r,F)$.

Proof. With λ, x^i_j as indeterminates, we take R to be the commutative ring $F[\lambda, x^i_j]$ with identity. Since $\det(X)$ is an invariant polynomial on $\mathfrak{gl}(r,F)$ (Example B.4), by Proposition B.5 for any $A \in \mathrm{GL}(r,F)$,

$$\begin{aligned}\det(\lambda I + X) &= \det(A^{-1}(\lambda I + X)A)\\ &= \det(\lambda I + A^{-1}XA).\end{aligned}$$

Hence,

$$\sum_{k=0}^{r} f_k(X)\lambda^{r-k} = \sum_{k=0}^{r} f_k(A^{-1}XA)\lambda^{r-k}.$$

Comparing the coefficients of λ^{r-k} on both sides gives

$$f_k(A^{-1}XA) = f_k(X) \quad \text{for all } A \in \mathrm{GL}(r,F).$$

This proves that all the polynomials $f_k(X)$ are invariant polynomials on $\mathfrak{gl}(r,F)$. $\square$

For each integer $k \geq 0$, the kth ***trace polynomial*** is defined to be

$$\Sigma_k(X) = \mathrm{tr}(X^k) \in \mathbb{Z}[x^i_j].$$

Since $\mathrm{tr}(X) = f_1(X)$ is an invariant polynomial on $\mathfrak{gl}(r,F)$, for any $A \in \mathrm{GL}(r,F)$,

$$\Sigma_k(A^{-1}XA) = \mathrm{tr}(A^{-1}X^kA) = \mathrm{tr}(X^k) = \Sigma_k(X).$$

So the trace polynomials $\Sigma_k(X)$ are also invariant polynomials on $\mathfrak{gl}(r,F)$.

B.4 Invariant Complex Polynomials

The goal of this subsection is to determine all complex polynomials on $\mathfrak{gl}(r,\mathbb{C})$ invariant under conjugation by elements of $\mathrm{GL}(r,\mathbb{C})$.

Suppose $X \in \mathfrak{gl}(r,\mathbb{C})$ is a diagonalizable matrix of complex numbers (as opposed to indeterminates). Because X is diagonalizable, there exists a nonsingular matrix $A \in \mathrm{GL}(r,\mathbb{C})$ such that

$$A^{-1}XA = \begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{bmatrix} = \mathrm{diag}(t_1,\dots,t_r),$$

with $t_1,\dots,t_r$ being the eigenvalues of X. For such a matrix X,

$$\mathrm{tr}(X) = \mathrm{tr}(A^{-1}XA) = \sum t_i,$$
$$\det(X) = \det(A^{-1}XA) = \prod t_i.$$

Thus, $\mathrm{tr}(X)$ and $\det(X)$ can be expressed as symmetric polynomials of their eigenvalues. This example suggests that to an invariant polynomial $P(X)$ on $\mathfrak{gl}(r,\mathbb{C})$, one can associate a symmetric polynomial $\tilde{P}(t_1,\dots,t_r)$ such that if $t_1,\dots,t_r$ are the eigenvalues of a complex matrix $X \in \mathfrak{gl}(r,\mathbb{C})$, then $P(X) = \tilde{P}(t_1,\dots,t_r)$. Note that $\sum t_i$ and $\prod t_i$ are simply the restriction of $\mathrm{tr}(X)$ and $\det(X)$, respectively, to the diagonal matrix $\mathrm{diag}(t_1,\dots,t_r)$.

Let $\mathrm{Inv}(\mathfrak{gl}(r,\mathbb{C}))$ be the algebra of all invariant complex polynomials on $\mathfrak{gl}(r,\mathbb{C})$, and $\mathbb{C}[t_1,\dots,t_r]^{S_r}$ the algebra of complex symmetric polynomials in $t_1,\dots,t_r$. To an invariant polynomial $P(X) \in \mathrm{Inv}(\mathfrak{gl}(r,\mathbb{C}))$, we associate the polynomial

$$\tilde{P}(t_1,\dots,t_r) = P\left(\begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{bmatrix}\right).$$

If $\sigma \in S_r$ is a permutation of $\{1,\dots,r\}$, the ***permutation matrix*** $P_\sigma \in \mathrm{GL}(r,\mathbb{Z})$ is the matrix whose $(i,\sigma(i))$-entry is 1 and whose other entries are all 0. For example,

$$P_{(12)} = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}.$$

If σ is a transposition, then P_σ is called a ***transposition matrix***.

It is easy to check that conjugating the diagonal matrix $\mathrm{diag}(t_1,\dots,t_r)$ by the transposition matrix $P_{(ij)}$ interchanges t_i and t_j in $\mathrm{diag}(t_1,\dots,t_r)$. Consequently, by conjugating $\mathrm{diag}(t_1,\dots,t_r)$ by a product of transposition matrices, we can make $t_1,\dots,t_r$ appear in any order. Thus, $\tilde{P}(t_1,\dots,t_r)$ is invariant under all permutations of $t_1,\dots,t_r$, i.e., it is a symmetric polynomial in $t_1,\dots,t_r$.

Theorem B.7. *The map*

$$\varphi\colon\ \mathrm{Inv}(\mathfrak{gl}(r,\mathbb{C})) \to \mathbb{C}[t_1,\dots,t_r]^{S_r}$$
$$P(X) \mapsto \tilde{P}(t_1,\dots,t_r) = P\left(\begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{bmatrix}\right)$$

given by restricting an invariant polynomial $P(X)$ *to the diagonal matrix* $\mathrm{diag}(t_1,\dots,t_r)$ *is an isomorphism of algebras over* $\mathbb{C}$.

The ***elementary symmetric polynomials*** in a set of variables $t_1,\dots,t_r$ are the polynomials

$$\sigma_0 = 1, \quad \sigma_1 = \sum t_i, \quad \sigma_2 = \sum_{i<j} t_i t_j,$$
$$\vdots$$
$$\sigma_k = \sum_{1\le i_1<i_2<\cdots<i_k\le r} t_{i_1}t_{i_2}\cdots t_{i_k},$$
$$\vdots$$
$$\sigma_r = t_1\cdots t_r.$$

Example B.8. For any indeterminate λ, let

$$P_\lambda(X) = \det(\lambda I + X) = \sum_{k=0}^{r} f_k(X)\lambda^{r-k}.$$

Then

$$\varphi(P_\lambda(X)) = P_\lambda\left(\begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{bmatrix}\right) = \det\left(\lambda I + \begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{bmatrix}\right)$$
$$= \prod_{i=1}^{r}(\lambda + t_i)$$
$$= \sum_{k=0}^{r} \sigma_k(t)\lambda^{r-k}. \tag{B.1}$$

Since φ is an algebra homomorphism,

$$\varphi(P_\lambda(X)) = \varphi\Big(\sum_k f_k(X)\lambda^{r-k}\Big) = \sum_k \varphi(f_k(X))\lambda^{r-k}. \tag{B.2}$$

Comparing the coefficients of λ^{r-k} in (B.1) and (B.2) gives

$$\varphi(f_k(X)) = \sigma_k(t).$$

The proof of Theorem B.7 depends on two facts from algebra:

(1) The set of all diagonalizable $r\times r$ complex matrices is a dense subset of $\mathfrak{gl}(r,\mathbb{C})$.
(2) (**Fundamental theorem of symmetric polynomials**) Every symmetric polynomial in $t_1,\dots,t_r$ is a polynomial in the elementary symmetric polynomials σ_1, ..., σ_r.

Proof (*of* (1)). Recall from linear algebra that matrices with distinct eigenvalues are diagonalizable. The eigenvalues of the matrix X are the roots of its characteristic polynomial $f_X(\lambda) = \det(\lambda I - X)$. The polynomial $f_X(\lambda)$ has multiple roots if and only if $f_X(\lambda)$ and its derivative $f_X'(\lambda)$ have a root in common. Two polynomials $f(\lambda)$ and $g(\lambda)$ have a root in common if and only if their resultant $R(f,g)$, which is a polynomial in the coefficients of f and g, is zero. Thus, the matrix X has repeated eigenvalues if and only if $R(f_X, f_X') = 0$. Note that the resultant $R(f_X, f_X')$ is a polynomial with coefficients that are functions of entries of X. Since $R(f_X, f_X') = 0$ defines a subset of codimension one in $\mathfrak{gl}(n,\mathbb{C})$, the set of complex matrices with distinct eigenvalues is dense in $\mathfrak{gl}(r,\mathbb{C})$. Hence, the set of diagonalize matrices in $\mathfrak{gl}(n,\mathbb{C})$ is dense.

Proof (*of* (2)). This is a standard theorem in algebra. See, for example, [13, Chap. IV, §6, Th. 6.1, p. 191]. □

Proof (*of Theorem B.7*). *Injectivity of* φ: By Proposition B.1, because $\mathbb{C}$ is an infinite field, complex polynomials may be identified with complex polynomial functions. So we will interpret elements of both $\mathrm{Inv}(\mathfrak{gl}(r,\mathbb{C}))$ and $\mathbb{C}[t_1,\dots,t_r]^{S_r}$ as polynomial functions on $\mathfrak{gl}(r,\mathbb{C})$ and $\mathbb{C}^r$, respectively.

Suppose the invariant polynomial $P(X)$ vanishes on all diagonal matrices $X \in \mathfrak{gl}(r,\mathbb{C})$. By the invariance of $P(X)$ under conjugation, $P(X)$ vanishes on all diagonalizable matrices $X \in \mathfrak{gl}(r,\mathbb{C})$. Since the subset of diagonalizable matrices is dense

in $\mathfrak{gl}(r,\mathbb{C})$, by continuity $P(X)$ vanishes on $\mathfrak{gl}(r,\mathbb{C})$. Therefore, $P(X)$ is the zero function and hence the zero polynomial on $\mathfrak{gl}(r,\mathbb{C})$.

Surjectivity of φ: We know from Example B.8 that the elementary symmetric polynomial $\sigma_k(t_1,\dots,t_r)$ is the image of the polynomial $f_k(X)$ under φ. By the fundamental theorem on symmetric polynomials, every element of $\mathbb{C}[t_1,\dots,t_r]^{S_r}$ is of the form $Q(\sigma_1,\dots,\sigma_r)$ for some polynomial Q. Then

$$\begin{aligned} Q(\sigma_1,\dots,\sigma_r) &= Q\left(\varphi(f_1(X)),\dots,\varphi(f_r(X))\right) \\ &= \varphi(Q(f_1(X),\dots,f_r(X)) = \varphi(P(X)), \end{aligned}$$

where $P(X) = Q(f_1(X),\dots,f_r(X)) \in \operatorname{Inv}(\mathfrak{gl}(r,\mathbb{C}))$. □

Example B.9. If $\Sigma_k(X) = \operatorname{tr}(X^k)$ are the trace polynomials, then

$$\tilde{\Sigma}_k(t_1,\dots,t_r) = \operatorname{tr}\begin{bmatrix} t_1^k & & \\ & \ddots & \\ & & t_r^k \end{bmatrix} = \sum t_i^k = s_k.$$

So the symmetric polynomials corresponding to the trace polynomials Σ_k are the power sums s_k.

Theorem B.10. *The ring of invariant complex polynomials on* $\mathfrak{gl}(r,\mathbb{C})$ *is generated as an algebra over* $\mathbb{C}$ *by the coefficients* $f_i(X)$ *of the characteristic polynomial* $\det(\lambda I + X)$. *Thus,*

$$\operatorname{Inv}(\mathfrak{gl}(r,\mathbb{C})) = \mathbb{C}[f_1(X),\dots,f_r(X)].$$

Proof. Under the isomorphism

$$\varphi\colon \operatorname{Inv}(\mathfrak{gl}(r,\mathbb{C})) \xrightarrow{\sim} \mathbb{C}[t_1,\dots,t_r]^{S_r},$$

the polynomials $f_k(X)$ correspond to the elementary symmetric polynomials $\sigma_k(t_1,\dots,t_r)$. Since $\mathbb{C}[t_1,\dots,t_r]^{S_r} = \mathbb{C}[\sigma_1,\dots,\sigma_r]$ by the fundamental theorem on symmetric polynomials, the isomorphism φ gives

$$\operatorname{Inv}(\mathfrak{gl}(r,\mathbb{C})) = \mathbb{C}[f_1(X),\dots,f_r(X)].$$ □

B.5 L-Polynomials, Todd Polynomials, and the Chern Character

The L-polynomials, Todd polynomials, and the Chern character are three families of polynomials of great importance in algebraic topology. All three are defined in terms of elementary symmetric polynomials.

Consider the formal power series

$$f(t) = \frac{\sqrt{t}}{\tanh\sqrt{t}} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \cdots.$$

Let $t_1,\dots,t_r$ be algebraically independent variables over $\mathbb{Q}$. Then the formal power series

$$\begin{aligned} F(t_1,\dots,t_r) &= \prod_{i=1}^{r} f(t_i) = \prod_{i=1}^{r} \left(1+\frac{1}{3}t_i - \frac{1}{45}t_i^2 + \cdots\right) \\ &= 1 + F_1(t_1,\dots,t_r) + F_2(t_1,\dots,t_r) + F_3(t_1,\dots,t_r) + \cdots \end{aligned}$$

is symmetric in $t_1,\dots,t_r$, so that for each n its homogeneous component $F_n(t_1,\dots,t_r)$ is a symmetric polynomial of degree n. By the fundamental theorem of symmetric polynomials,

$$F_n(t_1,\dots,t_r) = L_n(x_1,\dots,x_r),$$

where $x_1,\dots,x_r$ are the elementary symmetric polynomials in $t_1,\dots,t_r$. The polynomials $L_n(x_1,\dots,x_r)$ are called the ***L-polynomials***.. The L-polynomials appear in Hirzebruch's signature formula (26.4).

It is easy to work out the first few L-polynomials:

$$\begin{aligned} F_1(t_1,\dots,t_r) &= \frac{1}{3}\sum t_i = \frac{1}{3}x_1 = L_1(x_1), \\ F_2(t_1,\dots,t_r) &= -\frac{1}{45}\sum t_i^2 + \frac{1}{9}\sum_{i<j} t_i t_j \\ &= -\frac{1}{45}\Big(\big(\sum_i t_i\big)^2 - 2\sum_{i<j} t_i t_j\Big) + \frac{1}{9}\sum_{i<j} t_i t_j \\ &= -\frac{1}{45}\big(\sum_i t_i\big)^2 + \frac{7}{45}\sum_{i<j} t_i t_j \\ &= \frac{1}{45}(7x_2 - x_1^2) = L_2(x_1,x_2). \end{aligned}$$

Instead of $f(t) = \sqrt{t}/\tanh\sqrt{t}$, one may apply the same construction to any analytic function to obtain a sequence of homogeneous polynomials. For example, starting with the formal power series of

$$f(t) = \frac{t}{1-e^{-t}} = 1 + \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{120} + \cdots,$$

one forms the product

$$\begin{aligned} F(t_1,\dots,t_r) &= \prod_{i=1}^{r} f(t_i) \\ &= 1 + F_1(t_1) + F_2(t_1,t_2) + \cdots \\ &= 1 + T_1(x_1) + T_2(x_1,x_2) + \cdots. \end{aligned}$$

The polynomials $T_n(x_1,\dots,x_n)$, called the ***Todd Polynomials***, play a key role in the Hirzebruch–Riemann–Roch theorem (26.1).

Again, it is easy to work out the first few Todd polynomials:

$$\begin{aligned}
F_1(t_1,\dots,t_r) &= \frac{1}{2}\sum t_i = \frac{1}{2}x_1 = T_1(x_1)\\
F_2(t_1,\dots,t_r) &= \frac{1}{12}\sum t_i^2 + \frac{1}{4}\sum_{i<j} t_i t_j\\
&= \frac{1}{12}\big((\sum t_i)^2 - 2\sum_{i<j} t_i t_j + 3\sum_{i<j} t_i t_j\big)\\
&= \frac{1}{12}(x_1^2 + x_2) = T_2(x_1, x_2).
\end{aligned}$$

The Chern character is defined differently from the L-polynomials and the Todd polynomials. Instead of the product of $f(t_i)$, we take the sum:

$$\mathrm{ch}(x_1,\dots,x_n) = \sum_{i=1}^{n} e^{t_i}.$$

Since in each degree $\sum_{i=1}^{n} e^{t_i}$ is a symmetric polynomial in $t_1,\dots,t_n$, it can be expressed as a polynomial in the elementary symmetric functions $x_1,\dots,x_n$. Thus,

$$\mathrm{ch}(x_1,\dots,x_n) = n + \mathrm{ch}_1(x_1,\dots,x_n) + \mathrm{ch}_2(x_1,\dots,x_n) + \dots,$$

where

$$\mathrm{ch}_k(x_1,\dots,_n) = \sum_{i=1}^{n} \frac{t_i^k}{k!}.$$

Then

$$\begin{aligned}
\mathrm{ch}_1(x_1,\dots,x_n) &= \sum t_i = x_1,\\
\mathrm{ch}_2(x_1,\dots,x_n) &= \sum \frac{t_i^2}{2} = \frac{(\sum t_i)^2 - 2\sum_{i<j} t_i t_j}{2}\\
&= \frac{x_1^2 - x_2}{2}.
\end{aligned}$$

The ***Chern character*** of a vector bundle E is defined to be

$$\mathrm{ch}(E) = \mathrm{ch}(c_1,\dots,c_n),$$

where $c_1,\dots,c_n$ are the Chern classes of E. Formally, if $c(E) = \prod(1+t_i)$, then $\mathrm{ch}(E) = \sum e^{t_i}$.

B.6 Invariant Real Polynomials

In this subsection we will prove the analogue of Theorem B.10 for invariant real polynomials, that every invariant real polynomial on $\mathfrak{gl}(r,\mathbb{R})$ is a real polynomial in the elementary symmetric polynomials.

Proposition B.11. *If a real homogeneous polynomial $P(X)$ on $\mathfrak{gl}(r,\mathbb{R})$ is invariant under conjugation by $\mathrm{GL}(r,\mathbb{R})$, then it is invariant under conjugation by $\mathrm{GL}(r,\mathbb{C})$.*

Proof. By Cramer's rule, $A^{-1} = A^*/\det A$, where A^* is the transpose of the matrix of signed minors of A. Suppose $P(X)$ has degree k. Then the equation

$$P(AXA^{-1}) = P(AXA^*/\det A) = P(X)$$

is equivalent by homogeneity to

$$P(AXA^*) = (\det A)^k P(X),$$

or

$$q(A,X) := P(AXA^*) - (\det A)^k P(X) = 0. \tag{B.3}$$

This is a polynomial equation in a_{ij} and x_{ij} that vanishes for $(A,X) \in \mathrm{GL}(r,\mathbb{R}) \times \mathbb{R}^{n\times n}$. By continuity, (B.3) holds for all $(A,X) \in \mathbb{R}^{r\times r} \times \mathbb{R}^{n\times n}$. Since $q(A,X)$ gives a holomorphic function on $\mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}$, if it vanishes identically over $\mathbb{R}$, then it vanishes identically over $\mathbb{C}$. (This is a property of holomorphic functions. It can be proven by noting that a holomorphic function is expressible as a power series, and if the function vanishes identically over $\mathbb{R}$, then the power series over $\mathbb{R}$ is zero. Since the power series over $\mathbb{C}$ is the same as the power series over $\mathbb{R}$, the function is identically zero.) □

Proposition B.11 may be paraphrased by the equation

$$\mathrm{Inv}\left(\mathfrak{gl}(r,\mathbb{R})\right) = \mathrm{Inv}(\mathfrak{gl}(r,\mathbb{C})) \cap \mathbb{R}[x^i_j].$$

Theorem B.12. *Let $\varphi_{\mathbb{R}}$ be the restriction of the map φ of Theorem B.7 to* $\mathrm{Inv}\left(\mathfrak{gl}(r,\mathbb{R})\right)$*:*

$$\varphi_{\mathbb{R}}\colon \mathrm{Inv}\left(\mathfrak{gl}(r,\mathbb{R})\right) \to \mathbb{R}[t_1,\dots,t_r]^{S_r}$$
$$P(X) \mapsto \tilde{P}(t_1,\dots,t_r) = P\left(\begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{bmatrix}\right).$$

Then $\varphi_{\mathbb{R}}$ is an algebra isomorphism over $\mathbb{R}$.

Proof. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Inv}(\mathfrak{gl}(r,\mathbb{C})) & \xrightarrow{\varphi} & \mathbb{C}[t_1,\dots,t_r]^{S_r} \\ \cup & & \cup \\ \mathrm{Inv}\left(\mathfrak{gl}(r,\mathbb{R})\right) & \xrightarrow{\varphi_{\mathbb{R}}} & \mathbb{R}[t_1,\dots,t_r]^{S_r}. \end{array}$$

The injectivity of $\varphi_{\mathbb{R}}$ follows immediately from the injectivity of φ.

As for the surjectivity of $\varphi_{\mathbb{R}}$, the proof is the same as the complex case (Theorem B.7). More precisely, every symmetric polynomial $p(t_1,\dots,t_r) \in \mathbb{R}[t_1,\dots,t_r]^{S_r}$ is a real polynomial in the elementary symmetric polynomials $\sigma_1,\dots,\sigma_r$ of $t_1,\dots,t_r$:

$$p(t_1,\dots,t_r) = q(\sigma_1,\dots,\sigma_r).$$

Since $\sigma_k(t) = \varphi(f_k(X))$ and φ is an algebra homomorphism,

$$p(t_1,\dots,t_r) = q\left(\varphi(f_1(X)),\dots,\varphi(f_r(X))\right) = \varphi\left(q(f_1(X),\dots,f_r(X))\right),$$

which proves that $\varphi_{\mathbb{R}}$ is onto. □

Theorem B.13. *The ring of invariant real polynomials on* $\mathfrak{gl}(r,\mathbb{R})$ *is generated as an algebra over* $\mathbb{R}$ *by the coefficients* $f_i(X)$ *of the characteristic polynomial* $\det(\lambda I + X)$. *Thus,*

$$\mathrm{Inv}(\mathfrak{gl}(r,\mathbb{R})) = \mathbb{R}[f_1(X),\dots,f_r(X)].$$

Proof. The proof is the same as that of Theorem B.10, with $\mathbb{R}$ instead of $\mathbb{C}$. □

B.7 Newton's Identities

Among the symmetric polynomials in $t_1,\dots,t_r$, two sets are of special significance: the elementary symmetric polynomials $\sigma_1,\dots,\sigma_r$ and the power sums $s_1,\dots,s_r$. Newton's identities give relations among the two sets. As a corollary, each set generates the algebra of symmetric polynomials over $\mathbb{R}$. Define $\sigma_0 = 1$.

Theorem B.14 (Newton's identities). *For each integer* $k \geq 1$,

$$\left(\sum_{i=0}^{k-1}(-1)^i \sigma_i s_{k-i}\right) + (-1)^k k\sigma_k = 0,$$

or written out,

$$s_k - \sigma_1 s_{k-1} + \sigma_2 s_{k-2} - \cdots + (-1)^{k-1}\sigma_{k-1}s_1 + (-1)^k k\sigma_k = 0.$$

Proof (*from* [2, p. 79]). Consider the identity

$$(1-t_1x)\cdots(1-t_rx) = 1 - \sigma_1 x + \sigma_2 x^2 - \cdots + (-1)^r \sigma_r x^r.$$

Denote the right-hand side by $f(x)$ and take the logarithmic derivative of both sides to get

$$\ln(1-t_1x) + \cdots + \ln(1-t_rx) = \ln f(x),$$
$$\frac{-t_1}{1-t_1x} + \cdots + \frac{-t_r}{1-t_rx} = \frac{f'(x)}{f(x)}.$$

Hence,

$$\frac{t_1x}{1-t_1x} + \cdots + \frac{t_rx}{1-t_rx} = \frac{-xf'(x)}{f(x)}.$$

Expanding the left-hand side as a sum of geometric series, we get

$$\sum_{i=1}^{\infty}(t_1x)^i+\cdots+\sum_{i=1}^{\infty}(t_rx)^i=\frac{-xf'(x)}{f(x)},$$
$$\sum_{i=1}^{\infty}s_ix^i=\frac{-xf'(x)}{f(x)}.$$

Now multiply both sides by $f(x)$:

$$f(x)\left(\sum_{i=1}^{\infty}s_ix^i\right)=-xf'(x),$$
$$(1-\sigma_1x+\sigma_2x^2-\cdots+(-1)^r\sigma_rx^r)(s_1x+s_2x^2+\cdots)$$
$$=\sigma_1x-2\sigma_2x^2+\cdots+(-1)^{r+1}r\sigma_rx^r.$$

A comparison of the coefficients of x^k on both sides of the equation gives

$$s_k-\sigma_1s_{k-1}+\cdots+(-1)^{k-1}\sigma_{k-1}s_1=(-1)^{k+1}k\sigma_k.$$ □

Using Newton's identities we can write each elementary symmetric polynomial σ_k as a polynomial in the power sums.

Example B.15. For σ_1 and σ_2 we can get the relations directly:

$$\sigma_1=t_1+\cdots+t_r=s_1,$$
$$\sigma_2=\sum_{i<j}t_it_j=\frac{1}{2}((t_1+\cdots+t_r)^2-(t_1^2+\cdots+t_r^2))$$
$$=\frac{1}{2}\left(s_1^2-s_2\right).$$

For σ_3, Newton's identities give

$$3\sigma_3=s_3-\sigma_1s_2+\sigma_2s_1$$
$$=s_3-s_1s_2+\frac{1}{2}\left(s_1^2-s_2\right)s_1.$$

A mathematical induction on k using Newton's identities proves that every elementary symmetric polynomial σ_k is a polynomial in the power sums $s_1,\ldots,s_k$.

Theorem B.16. *The algebra of symmetric polynomials over* $\mathbb{R}$ *can be generated as an algebra over* $\mathbb{R}$ *by the elementary symmetric polynomials or by the power sums:*

$$\mathbb{R}[t_1,\ldots,t_r]^{S_r}=\mathbb{R}[\sigma_1,\ldots,\sigma_r]=\mathbb{R}[s_1,\ldots,s_r].$$

Proof. We have just shown that every σ_k is a polynomial in $s_1,\ldots,s_r$. Conversely, since each s_k is symmetric, by the fundamental theorem on symmetric polynomials it is a polynomial in $\sigma_1,\ldots,\sigma_r$. □

Theorem B.17. *The algebra of invariant polynomials on* $\mathfrak{gl}(r,\mathbb{R})$ *is generated as an algebra over* $\mathbb{R}$ *by the coefficients* $f_1(X),\ldots,f_r(X)$ *of the characteristic polynomial* $\det(\lambda I+X)$ *or by the trace polynomials* $\operatorname{tr}(X),\ldots,\operatorname{tr}(X^r)$.

Proof. By Examples B.8 and B.9, under the algebra isomorphism

$$\phi\colon \operatorname{Inv}\left(\mathfrak{gl}(r,\mathbb{R})\right) \to \mathbb{R}[t_1,\ldots,t_r]^{S_r},$$

the polynomials $f_1(X),\ldots,f_r(X)$ correspond to the generators $\sigma_1,\ldots,\sigma_r$ and the trace polynomials correspond to the generators $s_1,\ldots,s_r$. This theorem then follows from Theorems B.13 and B.16. □

An analogous statement holds for the complex invariant polynomials on $\mathfrak{gl}(r,\mathbb{C})$.

Problems

B.1. The invariant polynomial $f_2(X)$
Let $X = [x^i_j]$ be an $r\times r$ matrix of indeterminates. The polynomials $f_k(X)\in\mathbb{Z}[x^i_j]$ are the coefficients in the characteristic polynomial

$$\det(\lambda I+X) = \sum_{k=0}^{r} f_k(X)\lambda^{r-k}.$$

Find a formula for the polynomial $f_2(X)\in\mathbb{Z}[x^i_j]$. Write out all the terms of the formula when $r=3$.

Hints and Solutions to Selected End-of-Section Problems

Problems with complete solutions are starred (*). Equations are numbered consecutively within each problem.

1.1 Positive-definite symmetric matrix

Fixing a basis $e_1,\dots,e_n$ for V defines an isomorphism $V \xrightarrow{\sim} \mathbb{R}^n$ by mapping a vector in V to its coordinates relative to $e_1,\dots,e_n$. So we may work exclusively with column vectors in $\mathbb{R}^n$. Because A is positive-definite, $\langle \mathbf{x},\mathbf{x}\rangle = \mathbf{x}^T A\mathbf{x} \geq 0$ with equality if and only if $\mathbf{x} = 0$. Because A is symmetric,

$$\langle \mathbf{y},\mathbf{x}\rangle = \mathbf{y}^T A\mathbf{x} = (\mathbf{x}^T A^T \mathbf{y})^T = (\mathbf{x}^T A\mathbf{y})^T = \mathbf{x}^T A\mathbf{y} = \langle \mathbf{x},\mathbf{y}\rangle.$$

Finally, the expression $\mathbf{x}^T A\mathbf{y}$ is clearly linear in $\mathbf{x}$ and in $\mathbf{y}$. □

1.2* Inner product

By linearity in the first argument,

$$\langle u,w\rangle = \langle v,w\rangle \quad \text{iff} \quad \langle u-v,w\rangle = 0.$$

Since the inner product is positive-definite, this implies that $u-v=0$. □

1.4* Positive linear combination of inner products

For any $v \in V$, we have $\langle v,v\rangle = \sum a_i \langle v,v\rangle_i \geq 0$. If equality holds, then every term is 0 and therefore v=0 by the positive-definiteness of any of the inner products $\langle\ ,\ \rangle_i$. Symmetry and bilinearity in either argument are just as obvious. □

1.5* Extending a vector to a vector field

Let $(U,x^1,\dots,x^n)$ be a coordinate chart about p. If $Z_p = \sum a^i \partial/\partial x^i|_p$, then $Z := \sum a^i \partial/\partial x^i$ is a C^∞ vector field on U that extends Z_p. To extend Z_p to a C^∞ vector field on M, let f be a C^∞ bump function supported in U which is 1 at p. Define

$$X_q = \begin{cases} f(q)Z_q, & \text{for } q \in U, \\ 0, & \text{for } q \notin U. \end{cases}$$

Then X is a vector field on M such that $X_p = Z_p$.

L.W. Tu, *Differential Geometry*, Graduate Texts in Mathematics 275,
DOI 10.1007/978-3-319-55084-8

On U the vector field X is C^∞ because it is the product of a C^∞ function and a C^∞ vector field Z. If $q \notin U$, then since $\operatorname{supp} f$ is a closed subset $\subset U$, there is a neighborhood V of q such that $V \cap \operatorname{supp} f = \varnothing$. Thus, $X \equiv 0$ on V, and is trivially C^∞ at q. This proves that X is C^∞ at every point of M. □

1.6* **Equality of vector fields**
(⇒) If $X = Y$, then clearly $\langle X, Z\rangle = \langle Y, Z\rangle$ for all $Z \in \mathfrak{X}(M)$.
(⇐) To prove that $X = Y$, it suffices to show that $X_p = Y_p$ for all $p \in M$. Let Z_p be any vector in T_pM. By Problem 1.5, Z_p can be extended to a C^∞ vector field Z on M. By hypothesis, $\langle X, Z\rangle = \langle Y, Z\rangle$. Hence,

$$\langle X_p, Z_p\rangle = \langle X, Z\rangle_p = \langle Y, Z\rangle_p = \langle Y_p, Z_p\rangle.$$

By Problem 1.2, this implies that $X_p = Y_p$. Since p is an arbitrary point of M, it follows that $X = Y$. □

1.7* **Hyperbolic upper half-plane**
Since $1/y^2$ is always positive on $\mathbb{H}^2$, by Problem 1.4 the function $\langle\ ,\ \rangle'$ is an inner product at each point $p \in \mathbb{H}^2$. It remains to check that if $X, Y \in \mathfrak{X}(\mathbb{H}^2)$, then $\langle X, Y\rangle' = (1/y^2)\langle X, Y\rangle$ is a C^∞ function on $\mathbb{H}^2$. This is true because being the Euclidean metric, $\langle X, Y\rangle$ is C^∞ on $\mathbb{H}^2$ and $1/y^2$ is C^∞ on $\mathbb{H}^2$. □

2.1* **Signed curvature**
If $\gamma'(s) = T(s) = (\cos\theta(s), \sin\theta(s))$, then

$$\gamma''(s) = T'(s) = \begin{bmatrix} -\theta'(s)\sin\theta(s) \\ \theta'(s)\cos\theta(s) \end{bmatrix} \text{ and } \mathbf{n}(s) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta(s) \\ \sin\theta(s) \end{bmatrix} = \begin{bmatrix} -\sin\theta(s) \\ \cos\theta(s) \end{bmatrix}.$$

Therefore, $\gamma''(s) = \theta'(s)\mathbf{n}(s)$. This proves that $\kappa = \theta'(s)$. □

2.3 Write $\theta = \arctan(\dot{y}/\dot{x})$. Compute $\kappa = d\theta/ds = (d\theta/dt)/(ds/dt)$, where $ds/dt = (\dot{x}^2 + \dot{y}^2)^{1/2}$.

2.4 In Problem 2.3, set the parameter $t = x$.

2.5 $\kappa = ab/(a^2\sin^2 t + b^2\cos^2 t)$.

2.6 $(1/27)\big[-8 + (4 + 9a^2)^{3/2}\big]$.

2.7* **Curvature of a space curve**

a) Since

$$s(t) = \int_0^t |c'(u)|\, du = \int_0^t \sqrt{a^2+b^2}\, du = \sqrt{a^2+b^2}\, t,$$

$t = s/\sqrt{a^2+b^2}$.

b) Let $C = 1/\sqrt{a^2+b^2}$. Then $\gamma(s) = (a\cos Cs, a\sin Cs, bCs)$. A short computation gives $|\gamma''(s)| = a/(a^2+b^2)$. □

3.1 $\kappa_1, \kappa_2 = H \pm 2\sqrt{H^2 - K}$.

5.3* **The Gauss map**
Let $c(t)$ be a curve through p with $c'(0) = X_p$. Define a vector field $\bar{N}(t)$ along c by $\bar{N}(t) = dN_{c(t)}/dt$. Then

$$\begin{aligned} \nu_{*,p}(X_p) &= \frac{d}{dt}\bigg|_{t=0} \nu(c(t)) = \frac{d}{dt}\bigg|_{t=0} N_{c(t)} = \frac{d}{dt}\bigg|_{t=0} \bar{N}(t) \\ &= \bar{N}'(0) = D_{c'(0)}N \qquad \text{(by Proposition 4.11)} \\ &= D_{X_p}N = -L(X_p). \end{aligned}$$

□

5.4 Area $\nu(M) = \int_{\nu(M)} 1 = \int_M |\det \nu_*|$ by the change of variables formula. Now apply Problem 5.3.

5.8 Choose an orthonormal basis e_1, e_2 for T_pM and write $X_p = (\cos\theta)e_1 + (\sin\theta)e_2$. Let the shape operator L be represented by the matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ relative to e_1, e_2. Find $\kappa(X_p)$ in terms of a, b, c, d and θ and compute $\int_0^{2\pi} \kappa(X_p)\,d\theta$.

5.9 (a) Take the inner product of both sides of the equation $L(e_1) = ae_1 + be_2$ with e_1 and with e_2. Then solve for a, b.

(b) $K = \det \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \det \begin{bmatrix} e & f \\ f & g \end{bmatrix} = (eg - f^2)/(EG - F^2)$.
$H = (eG - 2fF + gE)/\big(2(EG - F^2)\big)$.

5.10 (a) First show that $e_1 = (1, 0, h_x)$, $e_2 = (0, 1, h_y)$.

(b) By Lemma 5.2, $\langle L(e_i), e_j \rangle = \langle D_{e_i} e_j, N \rangle$, where $N = (e_1 \times e_2)/\|e_1 \times e_2\|$. Show that

$$D_{e_1} e_1 = (0, 0, h_{xx}), \quad D_{e_2} e_1 = (0, 0, h_{xy}), \quad D_{e_2} e_2 = (0, 0, h_{yy}).$$

(c) Use Problem 5.9(b).

5.11 (a) $e_1 = \big((\sinh u)\cos\theta, (\sinh u)\sin\theta, 1\big)$, $e_2 = \big(-(\cosh u)\sin\theta, (\cosh u)\cos\theta, 0\big)$.

(b) $E = \cosh^2 u$, $F = 0$, $G = \cosh^2 u$.
(c) $e_1' = \big((\cosh u)\cos\theta, (\cosh u)\sin\theta, 0\big)$, $e_2' = \big(-(\sinh u)\sin\theta, (\sinh u)\cos\theta, 1\big)$.

7.1* **Derivation and local operator**
Suppose a C^∞ function g on M is zero on an open set U. Let $p \in U$ and choose a bump function $f \in C^\infty(M)$ such that $f(p) = 1$ and $\operatorname{supp} f \subset U$. then $fg \equiv 0$ because $g \equiv 0$ on U and $f \equiv 0$ on $M - U$. By the derivation property,

$$0 = (D(fg))(p) = (Df)(p)g(p) + f(p)(Dg)(p) = Dg(p),$$

since $g(p) = 0$ and $f(p) = 1$. As p is an arbitrary point of U, $Dg \equiv 0$ on U. □

7.4* **Section with a prescribed value**
Let U be a neighborhood of p in M over which E is trivial: $E|_U \overset{\phi}{\simeq} U \times \mathbb{R}^k$. Suppose $\phi(e) = (p, v)$. Then $s_U(q) := \phi^{-1}(q, v)$ is a C^∞ section of E over U with $s_U(p) = \phi^{-1}(p, v) = e$. By Proposition 7.13, there is a C^∞ global section $s \in \Gamma(M, E)$ that agrees with s_U over some neighborhood of p. Then $s(p) = s_U(p) = e$. □

7.5* **Coefficients relative to a global frame**
It suffices to show that a^j is C^∞ over any trivializing open set for E. Choose a trivializing open set U for E with trivialization

$$\phi : \pi^{-1}(U) \simeq U \times \mathbb{R}^k, \qquad e \mapsto (p, b^1, \ldots, b^k).$$

Let $s_1, \dots, s_k$ be the frame over U corresponding to the standard basis in $\mathbb{R}^k$ under ϕ. Then for any $v \in \pi^{-1}(U)$, $v = \sum b^i s_i$. The b^i's are C^∞ functions on E because they are components of the C^∞ map ϕ.

Since $e_1, \dots, e_k$ is a C^∞ frame over U,

$$e_j = \sum f_j^i s_i$$

with f_j^i C^∞. Then

$$v = \sum b^i s_i = \sum a^j e_j = \sum a^j f_j^i s_i.$$

Comparing the coefficients of s_i gives

$$b^i = \sum_j a^j f_j^i.$$

In matrix notation, $\mathbf{b} = C\mathbf{a}$. Hence, $\mathbf{a} = C^{-1}\mathbf{b}$, which shows that the a^j are C^∞ because the entries of C and $\mathbf{b}$ are all C^∞. □

12.1* The orthogonal group $\mathrm{O}(2)$

(a) The condition $A^T A = I$ is equivalent to the fact that the columns of the matrix A form an orthonormal basis of $\mathbb{R}^2$. Let $\begin{bmatrix} a \\ b \end{bmatrix}$ be the first column of A. Then $a^2 + b^2 = 1$. Since the second column of A is a unit vector in $\mathbb{R}^2$ orthogonal to $\begin{bmatrix} a \\ b \end{bmatrix}$, it is either $\begin{bmatrix} -b \\ a \end{bmatrix}$ or $\begin{bmatrix} b \\ -a \end{bmatrix}$. Hence,

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{ or } \begin{bmatrix} a & b \\ b & -a \end{bmatrix},$$

with $a^2 + b^2 = 1$.

(b) In part (a),

$$\det \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a^2 + b^2 = 1, \quad \det \begin{bmatrix} a & b \\ b & -a \end{bmatrix} = -a^2 - b^2 = -1.$$

Therefore,

$$\mathrm{SO}(2) = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \;\middle|\; a^2 + b^2 = 1 \right\}.$$

The condition $a^2 + b^2 = 1$ implies that (a,b) is a point on the unit circle. Hence, (a,b) is of the form $(\cos t, \sin t)$ for some $t \in \mathbb{R}$. So every element of $\mathrm{SO}(2)$ is of the form

$$\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

□

14.1* Geodesic equations

Differentiating (14.10) with respect to t gives

$$\begin{aligned} 2\ddot{y} &= \pm \dot{y}\sqrt{1 - k^2 y^2} \mp y \frac{k^2 y \dot{y}}{\sqrt{1 - k^2 y^2}} \\ &= \pm \dot{y}\left(\pm \frac{\dot{y}}{y}\right) \mp \frac{k^2 y^3 \dot{y}}{y\sqrt{1 - k^2 y^2}} = \frac{\dot{y}^2}{y} - k^2 y^3 && \text{(by (14.8))} \\ &= \frac{\dot{y}^2}{y} - \frac{k^2 y^4}{y} = \frac{\dot{y}^2}{y} - \frac{\dot{x}^2}{y}. && \text{(by (14.10))} \end{aligned}$$

Thus,

$$\ddot{y} + \frac{1}{y}\dot{x}^2 - \frac{1}{y}\dot{y}^2 = 0,$$

which is (14.5). □

16.4* Volume form of a sphere in Cartesian coordinates

The unit outward normal vector field on the sphere $S^{n-1}(a)$ is $X = (1/a)\sum x^i \partial/\partial x^i$. The volume form on the closed ball $\bar{B}^n(a)$ of radius a is the restriction of the volume form on $\mathbb{R}^n$; thus,

$$\mathrm{vol}_{\bar{B}^n(a)} = dx^1 \wedge \cdots \wedge dx^n.$$

By Theorem 16.11,

$$\begin{aligned}\mathrm{vol}_{S^{n-1}(a)} &= \iota_X (dx^1 \wedge \cdots \wedge dx^n) \\ &= \frac{1}{a}\sum x^i \iota_{\partial/\partial x^i}(dx^1 \wedge \cdots \wedge dx^n) \\ &= \frac{1}{a}\sum (-1)^{i-1} x^i \, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n. \end{aligned}$$ □

20.1* Topology of a union

Using the bijection $\phi_\alpha : S_\alpha \to Y_\alpha$ in (iii), each S_α can be given a topology so that ϕ_α becomes a homeomorphism. Condition (iv) guarantees that the open subsets of $S_\alpha \cap S_\beta$ are well defined. Let $\mathcal{T}$ be the topology on S generated by all the open subsets of S_α for all α; this means $\mathcal{T}$ is the smallest topology containing all such sets.

If U_α is open in S_α and U_β is open in S_β, then $U_\alpha \cap U_\beta$ is open in $\mathcal{T}$. Since $U_\alpha \cap U_\beta \subset S_\alpha$, we need to show that $U_\alpha \cap U_\beta$ is already open in S_α so that in generating $\mathcal{T}$ we did not create additional open sets in S_α. By condition (ii), $S_\alpha \cap S_\beta := S_{\alpha\beta}$ is in the collection $\mathcal{C}$, so it is open via $\phi_{\alpha\beta} : S_{\alpha\beta} \to Y_{\alpha\beta}$. By the compatibility of $S_{\alpha\beta}$ and S_α in (ii), $S_{\alpha\beta}$ is open S_α. Since U_α is open in S_α, $U_\alpha \cap S_{\alpha\beta} = U_\alpha \cap S_\beta$ is open in S_α. By (iv), $U_\alpha \cap S_\beta$ is also open in S_β. Therefore, $(U_\alpha \cap S_\beta) \cap U_\beta = U_\alpha \cap U_\beta$ is open in S_β. By (iv) again, it is also open in S_α. □

21.9 * The Maurer–Cartan equation

Since $d\theta + \frac{1}{2}[\theta, \theta]$ is a 2-form and therefore is bilinear over the C^∞ functions, it suffices to check the equation on a frame of vector fields. Such a frame is, for example, given by left-invariant vector fields. Thus, it suffices to check

$$d\theta(X,Y) + \frac{1}{2}[\theta, \theta](X,Y) = 0$$

for two left-invariant vector fields X, Y on G.

By Problem 21.8,

$$d\theta(X,Y) = X\theta(Y) - Y\theta(X) - \theta\big([X,Y]\big).$$

Since $\theta(X)$ and $\theta(Y)$ are constants, $X\theta(Y) = 0$ and $Y\theta(X) = 0$. Thus,

$$d\theta(X,Y) = -\theta([X,Y]) = -\theta_e\big([X,Y]_e\big) = -[X,Y]_e = -[X_e, Y_e].$$

By (21.10),

$$\begin{aligned}[\theta,\theta](X,Y) &= [\theta(X),\theta(Y)] - [\theta(Y),\theta(X)]\\ &= 2[\theta(X),\theta(Y)]\\ &= 2[\theta_e(X_e),\theta_e(Y_e)] \quad \text{since } \theta(X),\theta(Y) \text{ are constants)}\\ &= 2[X_e,Y_e].\end{aligned}$$

The Maurer–Cartan equation follows. □

22.4 If T_1 is an (a,b)-tensor field and T_2 is an (a',b')-tensor field, evaluate both sides on $\omega_1,\dots,\omega_{a+a'},Y_1,\dots,Y_{b+b'}$.

27.2* **Short exact sequence of vector spaces**
By the first isomorphism theorem of linear algebra,

$$\frac{B}{i(A)} = \frac{B}{\ker j} \simeq \operatorname{im} j = C.$$

Hence,

$$\dim B - \dim i(A) = \dim C.$$

Because $i\colon A \to B$ is injective, $\dim i(A) = \dim A$. Therefore,

$$\dim B = \dim A + \dim C.$$ □

27.4* **The differential of an action**
By linearity,

$$\mu_*(X_p,\ell_{g*}A) = \mu_*(X_p,0) + \mu_*(0,\ell_{g*}A),$$

so it suffices to compute the two terms on the right separately.

Let $c(t)$ be a curve through p in P with initial vector X_p. Then a curve through (p,g) in $P\times G$ with initial vector $(X_p,0)$ is $\big(c(t),g\big)$. Computing the differential μ_* using the curve $\big(c(t),g\big)$, we get

$$\begin{aligned}\mu_{*,(p,g)}(X_p,0) &= \frac{d}{dt}\bigg|_{t=0} \mu\big(c(t),g\big)\\ &= \frac{d}{dt}\bigg|_{t=0} c(t)\cdot g\\ &= r_{g*}c'(0) = r_{g*}X_p.\end{aligned}$$

A curve through (p,g) with initial vector $(0,\ell_{g*}A)$ is (p,ge^{tA}), so

$$\begin{aligned}\mu_{*,(p,g)}(0,\ell_{g*}A) &= \frac{d}{dt}\bigg|_{t=0} \mu(p,ge^{tA})\\ &= \frac{d}{dt}\bigg|_{t=0} pge^{tA}\\ &= \underline{A}_{pg}.\end{aligned}$$

Hence,

$$\mu_{*,(p,g)}(X_p,\ell_{g*}A) = r_{g*}X_p + \underline{A}_p g.$$ □

30.3* **Lie bracket of horizontal vector fields**

$$\begin{aligned}\Omega(X,Y) &= \left(d\omega + \frac{1}{2}[\omega,\omega]\right)(X,Y)\\ &= (d\omega)(X,Y) + \frac{1}{2}\big([\omega(X),\omega(Y)] - [\omega(Y),\omega(X)]\big)\\ &= (d\omega)(X,Y) \quad \text{(since } X,\ Y \text{ are horizontal)}\\ &= X\omega(Y) - Y\omega(X) - \omega\big([X,Y]\big)\\ &= -\omega\big([X,Y]\big).\end{aligned}$$

Thus, $[X,Y]$ is horizontal if and only if $\Omega(X,Y)=0$. □

31.4* **Tensorial forms**

Let $p \in P$, $g \in G$, and $u_1,\dots,u_k \in T_pP$. Then

$$\begin{aligned}r_g^*(\psi_{pg}^\sharp)(u_1,\dots,u_k) &= \psi_{pg}^\sharp(r_{g*}u_1,\dots,r_{g*}u_k)\\ &= f_{pg}^{-1}(\psi_x(\pi_* r_{g*}u_1,\dots,\pi_* r_{g*}u_k))\\ &= \rho(g^{-1})f_p^{-1}(\psi_x(\pi_*u_1,\dots,\pi_*u_k)) \quad (\text{because } f_{pg} = f_p\circ\rho(g))\\ &= \rho(g^{-1})\psi_p^\sharp(u_1,\dots,u_k).\end{aligned}$$

□

31.5* **Tensorial forms**

Let $p \in P$ and $u_1,\dots,u_k \in T_pP$. Write $x = \pi(p)$. Then

$$\begin{aligned}(\varphi^{\flat\sharp})_p(u_1,\dots,u_k) &= f_p^{-1}(\varphi_x^\flat(\pi_*u_1,\dots,\pi_*u_k))\\ &= f_p^{-1}f_p\varphi_p(u_1,\dots,u_k)\\ &= \varphi_p(u_1,\dots,u_k).\end{aligned}$$

□

32.1* **Polynomials on a vector space**

Let $\alpha^1,\dots,\alpha^n$ be dual to $e_1,\dots,e_n$ and $\beta^1,\dots,\beta^n$ be dual to $u_1,\dots,u_n$. Then $\alpha^i = \sum c^i_j\beta^j$ for some invertible matrix $[c^i_j]$, and

$$\begin{aligned}f &= \sum a_I\alpha^{i_1}\cdots\alpha^{i_k}\\ &= \sum a_I c^{i_1}_{j_1}\cdots c^{i_k}_{j_k}\beta^{i_1}\cdots\beta^{i_k}.\end{aligned}$$

So f is also a polynomial of degree k with respect to $u_1,\dots,u_n$. □

32.2* **Chern–Weil forms**

Suppose $\alpha^i = \sum c^i_j\beta^j$. Then

$$\Omega^i = \alpha^i(\Omega) = \sum c^i_j\beta^j(\Omega) = \sum c^i_j\Psi^j$$

and

$$\begin{aligned}f &= \sum a_I c^{i_1}_{j_1}\beta^{j_1}\cdots c^{i_k}_{j_k}\beta^{j_k}\\ &= \sum a_I c^{i_1}_{j_1}\cdots c^{i_k}_{j_k}\beta^{j_1}\cdots\beta^{j_i}.\end{aligned}$$

So

$$\begin{aligned}f(\Psi) &= \sum a_I c^{i_1}_{j_1}\cdots c^{i_k}_{j_k}\Psi^{j_1}\cdots\Psi^{j_i}\\ &= \sum a_I\Omega^{j_1}\cdots\Omega^{j_i}\\ &= f(\Omega).\end{aligned}$$

□

List of Notations

$\mathbb{R}^n$	Euclidean n-space (p. 2)
(u^1, u^2, u^3)	a point in $\mathbb{R}^3$ (p. 2)
$\mathbf{u} = \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$	column vector (p. 2)
$\langle\ ,\ \rangle$	dot product, inner product, or Riemannian metric (p. 2)
$\lVert v \rVert$	length of a vector v (p. 2)
$\langle\ ,\ \rangle\vert_{W\times W}$	restriction of $\langle\ ,\rangle$ to a subspace W (p. 3)
A^T	transpose of the matrix A (p. 4)
T_pM	tangent space at p to M (p. 4)
$\mathbb{H}^2$	upper half-plane (p. 7)
$[a,b]$	closed interval from a to b (p. 9)
$T(s)$	unit tangent vector at time s (p. 11)
$T'(s)$	derivative of $T(s)$ (p. 11)
$\mathbf{n}(s)$	unit normal vector to a curve at time s (p. 11)
κ	signed curvature of a plane curve (p. 12)
$\dot{x}$	dx/dt (p. 14)
$\ddot{x}$	d^2x/dt^2 (p. 14)
N_p	normal vector at p (p. 17)
$\kappa(X_p)$	normal curvature (p. 18)
H	mean curvature (p. 18)
K	Gaussian curvature (p. 18)
$\chi(M)$	Euler characteristic (p. 20)
$D_{X_p}f$	directional derivative of f in the direction X_p (p. 22)

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∂_i	partial derivative $\partial/\partial x^i$ (p. 22)
$\mathfrak{X}(M)$	Lie algebra of all C^∞ vector fields on a manifold M (p. 23)
$[X,Y]$	Lie bracket of two vector fields (p. 24)
$T(X,Y)$	torsion tensor (p. 24)
$\mathrm{End}_{\mathbb{R}}(V)$	ring of endomorphisms over $\mathbb{R}$ of a vector space V (p. 24)
$R(X,Y)$	curvature tensor (p. 24)
$V(t)$	vector field along a curve (p. 25)
$c'(t)$	velocity vector field of a curve (p. 25)
$\partial_i\|_{c(t)}$	the vector field ∂_i evaluated at $c(t)$ (p. 26)
$\Gamma(TR\|_M)$	the set of C^∞ vector fields along a submanifold M in a manifold R (p. 26)
$Z(f)$	zero set of a function f (p. 29)
L_p	shape operator, Weingarten map (p. 30)
$\mathrm{II}(X_p,Y_p)$	second fundamental form (p. 35)
$\operatorname{grad} f$	gradient of a function f (p. 34)
ν	Gauss map (p. 39)
∇	affine connection, connection on a vector bundle (p. 43)
TM	tangent bundle of M (p. 51)
$\Gamma(U,E)$	the set of smooth sections of E over U (p. 51)
$\Gamma(E)$	the set of smooth global sections of E (p. 51)
$C^\infty(M)$	the algebra of C^∞ functions over M (p. 52)
$\Omega^k(M)$	space of C^∞ k-forms on M (p. 54)
$X_{\tan}$	tangential component of a vector field X along a manifold M (p. 62)
$\mathrm{III}(X,Y)$	third fundamental form (p. 68)
$\omega = [\omega^i_j]$	connection matrix (p. 80)
$\Omega = [\Omega^i_j]$	curvature matrix (p. 80)
$\alpha \wedge \beta$	wedge product of differential forms (p. 80)
$A \wedge B$	wedge product of matrices of differential forms (p. 86)
$\mathrm{proj}_a b$	orthogonal projection of b to the linear span of the vector a (p. 81)
θ^i	dual 1-form (p. 84)
τ^i	torsion form (p. 84)
$\mathbb{D}$	Poincaré disk (p. 86)
$K(P)$	sectional curvature of a plane P (p. 92)

$T(V)$	tensor algebra of a module V (p. 163)
$\bigwedge(V)$	exterior algebra of V (p. 164)
$\bigwedge^k(V)$, $\bigwedge^k V$	kth exterior power of V (p. 164)
$\mathrm{sgn}(\pi)$	sign of a permutation (p. 166)
$[s_1 \ \cdots \ s_k]$	a k-frame (p. 174)
f^*E	pullback bundle of E by f (p. 177)
$E\|_U$	restriction of the bundle E to U (p. 178)
$E \oplus E'$	direct sum of vector bundles (p. 181)
$E \otimes E'$	tensor product of vector bundles (p. 183)
$\mathrm{Iso}(V,V')$	the set of all isomorphisms from V to V' (p. 183)
$\bigwedge^k E$	kth exterior power of a vector bundle (p. 184)
$\Omega^k(M)$	vector space of C^∞ k-forms on M (p. 186)
$A_k(T,V)$	vector space of all alternating k-linear maps from T to V (p. 187)
$\Omega^k(M,V)$	vector space of smooth V-valued k-forms on M (p. 187)
$X_p f$	directional derivative of a vector-valued function f (p. 190)
$\alpha \cdot \beta$	product of vector-valued forms (p. 191)
$[\alpha, \beta]$	Lie bracket of $\mathfrak{g}$-valued forms (p. 191)
e_{ij}	the matrix with a 1 in the (i,j)-entry and 0 everywhere else (p. 193)
$\Omega^k(M,E)$	vector space of smooth E-valued k-forms on M (p. 194)
Ω^k	the wedge product $\Omega \wedge \cdots \wedge \Omega$ of the curvature matrix k times (p. 204)
tr	trace of a linear endomorphism (p. 209)
$\mathrm{Ric}(X,Y)$	Ricci curvature (p. 209)
$S(p)$	scalar curvature at p (p. 209)
$\mathrm{Inv}\big(\mathfrak{gl}(r,\mathbb{R})\big)$	algebra of invariant polynomials on $\mathfrak{gl}(r,\mathbb{R})$ (p. 212)
f_k	a coefficient of $\det(\lambda I + X)$ (p. 213)
$\Sigma_k(X)$	the trace polynomial $\mathrm{tr}(X^k)$ (p. 213)
$[P(\Omega)]$	the cohomology class of the form $P(\Omega)$ (p. 221)
$p_k(E)$	Pontrjagin class of a real vector bundle E (p. 225)
$\mathrm{SO}(r)$	special orthogonal group (p. 229)
$\mathfrak{so}(r)$	Lie algebra of $\mathrm{SO}(r)$ (p. 230)
$\mathrm{Pf}(X)$	Pfaffian of a skew-symmetric matrix X (p. 231)
$e(E)$	Euler class of an oriented Riemannian bundle E (p. 233)

$c_k(E)$	Chern class of a complex vector bundle (p. 235)
$\mathrm{Stab}(x)$	stabilizer of x (p. 243)
$\mathrm{Orbit}(x)$, xG	orbit of x (p. 243)
$U_{\alpha\beta}$	the intersection $U_\alpha \cap U_\beta$ (p. 245)
$\mathrm{Fr}(V)$	frame manifold of the vector space V (p. 246)
$\mathrm{Fr}(E)$	frame bundle of the vector bundle E (p. 246)
$\underline{A}_p$	fundamental vector field associated to A (p. 247)
$\mathcal{V}_p$	vertical tangent subspace (p. 250)
$\varphi_{a,b}$	parallel translation from $E_{c(a)}$ to $E_{c(b)}$ (p. 263)
$\mathcal{H}_p$	horizontal subspace (p. 266)
$v(Y_p)$	vertical component of a vector Y_p (p. 254)
$h(Y_p)$	horizontal component of a vector Y_p (p. 256)
$P \times_\rho V$	associated bundle (p. 275)
$\Omega^k_\rho(P,V)$	tensorial V-valued k-forms of type ρ (p. 277)
$\mathrm{Ad}\, P$	adjoint bundle (p. 279)
φ^h	horizontal component of a form φ (p. 281)
$D\varphi$	covariant derivative of a V-valued form on a principal bundle (p. 281)
$\mathrm{Inv}(\mathfrak{g})$	algebra of $\mathrm{Ad}(G)$-invariant polynomials on $\mathfrak{g}$ (p. 290)
(U,ϕ)	coordinate chart (p. 293)
$C^\infty_p(M)$	set of germs of C^∞ functions at p in M (p. 295)
$\mathbb{R}^2/\mathbb{Z}^2$	torus (p. 304)
$R[x_1,\dots,x_n]$	polynomial ring over R in n variables (p. 306)
$\mathrm{Fun}(\mathbb{R}^n,\mathbb{R})$	algebra of functions from $\mathbb{R}^n$ to $\mathbb{R}$ (p. 306)
$\mathbb{C}[t_1,\dots,t_r]^{S_r}$	the algebra of complex symmetric polynomials (p. 310)
P_σ	permutation matrix (p. 311)
σ_i	elementary symmetric polynomial (p. 311)

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